Answers to Selected Exercises

12. Finite Sampling Models

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1. Introduction

1.10.

a. {(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4), (4, 1), (4, 2), (4, 3), (4, 4)}
b. {(1, 2), (1, 3), (1, 4), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (3, 4), (4, 1), (4, 2), (4, 3)}
c. {{1, 1}, {1, 2}, {1, 3}, {1, 4}, {2, 2}, {2, 3}, {2, 4}, {3, 3}, {3, 4}, {4, 4}}
d. {{1, 2}, {1, 3}, {1, 4}, {2, 3}, {2, 4}, {3, 4}}

1.11. 0.4162
1.12.

a. 0.2070
b. 0.1859

1.14.

a. 0.9956
b. No

1.15. 0.2635
1.16. 0.1633.
1.23.

a. $\frac{1}{4}$
b. $\frac{1}{221}$
c. $\frac{4}{17}$
d. $\frac{1}{52}$

1.25. 0.000547
1.26.

a. $\frac{1}{1296}$

b. $\frac{5}{54}$
c. $\frac{5}{3888}$

1.28.

a. 0.2937
b. 0.7063

1.30.

a. $\frac{189}{625}$
b. $\frac{1}{10000}$

1.31.

a. $\frac{1}{25}$
b. $\frac{12}{25}$

2. The Hypergeometric Distribution

2.28. Let Y denote the number of defective chips in the sample

a. $\mathbb{P}(Y = k) = \frac{\binom{10}{k}\binom{90}{5-k}}{\binom{100}{5}}$ for $k \in \{0, 1, 2, 3, 4, 5\}$
b. $\mathbb{E}(Y) = 0.5$, $\text{var}(Y) = 0.432$,
c. $\mathbb{P}(Y > 0) = 0.416$,

2.29. Let Y denote the number of women, so that $Z = 10 - Y$, is the number of men.

a. $\mathbb{P}(Y = k) = \frac{\binom{30}{k}\binom{20}{10-k}}{\binom{50}{10}}$ for $k \in \{0, 1, ..., 10\}$
b. $\mathbb{E}(Y) = 6$, $\text{var}(Y) = 1.959$,
c. $\mathbb{E}(Z) = 4$, $\text{var}(Z) = 1.959$,
d. $\mathbb{P}(Y = 0) + \mathbb{P}(Y = 10) = 0.00294$,

2.30. Y denote the number of tagged fish in the sample

a. $\mathbb{P}(Y = k) = \frac{\binom{100}{k}\binom{900}{20-k}}{\binom{1000}{20}}$ for $k \in \{0, 1, ..., 20\}$
b. $\mathbb{E}(Y) = 2$, $\text{var}(Y) = \frac{196}{111}$
c. $\mathbb{P}(Y \geq 2) = 0.6108$,
d. $\mathbb{P}(Y \geq 2) \approx 0.6083$

2.31.

a. $\mathbb{P}(Y = k) = \binom{10}{k} 0.4^k \, 0.6^{10-k}$ for $k \in \{0, 1, ..., 10\}$.
b. $\mathbb{E}(Y) = 4$, $\text{var}(Y) = 2.4$
c. $\mathbb{P}(Y \geq 5) = 0.3669$

2.32. 20
2.33. 2000
2.34. 2000
2.35. Let U denote the number of spades and V the number of aces.

a. $\mathbb{P}(U = k) = \frac{\binom{13}{k}\binom{39}{5-k}}{\binom{52}{5}}$ for $k \in \{0, 1, ..., 5\}$, $\mathbb{E}(U) = \frac{5}{4}$, $\text{var}(U) = \frac{235}{272}$
b. $\mathbb{P}(V = k) = \frac{\binom{4}{k}\binom{48}{5-k}}{\binom{52}{5}}$ for $k \in \{0, 1, ..., 4\}$, $\mathbb{E}(V) = \frac{5}{13}$, $\text{var}(V) = \frac{940}{2873}$

2.36. Let U denote the number of hearts and V the number of honor cards.

a. $\mathbb{P}(U = k) = \frac{\binom{13}{k}\binom{39}{13-k}}{\binom{52}{13}}$ for $k \in \{0, 1, ..., 13\}$, $\mathbb{E}(U) = \frac{13}{4}$, $\text{var}(U) = \frac{507}{272}$
b. $\mathbb{P}(V = k) = \frac{\binom{16}{k}\binom{36}{13-k}}{\binom{52}{13}}$ for $k \in \{0, 1, ..., 13\}$, $\mathbb{E}(V) = 4$, $\text{var}(V) = \frac{36}{17}$

3. The Multivariate Hypergeometric Distribution

3.15. Let X, Y, and Z denote the number of republicans, democrats, and independents, respectively, in the sample.

a. $\mathbb{P}(X = i, Y = j, Z = k) = \frac{\binom{40}{i}\binom{35}{j}\binom{25}{k}}{\binom{100}{10}}$ where i, j, and k are nonnegative integers with $i + j + k = 10$.
b. $\mathbb{E}(X) = 4$, $\mathbb{E}(Y) = 3.5$, $\mathbb{E}(Z) = 2.5$
c. $\text{var}(X) = 2.1818$, $\text{var}(Y) = 2.0682$, $\text{var}(Z) = 1.7045$
d. $\text{cov}(X, Y) = -1.6364$, $\text{cov}(X, Z) = -0.9091$, $\text{cov}(Y, Z) = -0.7955$
e. 0.2474

3.16. Let X, Y, Z, U, and V denote the number of spades, hearts, diamonds, red cards, and black cards, respectively, in the hand.

a. $\mathbb{P}(X = i, Y = j, Z = k) = \frac{\binom{13}{i}\binom{13}{j}\binom{13}{k}\binom{13}{13-i-j-k}}{\binom{52}{13}}$ where i, j, and k are nonnegative integers with $i + j + k \leq 13$.
b. $\mathbb{P}(X = i, Y = j) = \frac{\binom{13}{i}\binom{13}{j}\binom{26}{13-i-j}}{\binom{52}{13}}$ where i and j are nonnegative integers with $i + j \leq 13$.
c. $\mathbb{P}(X = i) = \frac{\binom{13}{i}\binom{39}{13-i}}{\binom{52}{13}}$ for $i \in \{0, 1, ..., 13\}$.
d. $\mathbb{P}(U = i, V = j) = \frac{\binom{26}{i}\binom{26}{j}}{\binom{52}{13}}$ where i and j are nonnegative integers with $i + j = 13$.

3.17. Let X, Y, and U denote the number of spades, hearts, and red cards, respectively, in the hand.

a. $\mathbb{E}(X) = \frac{13}{4}$, $\text{var}(X) = \frac{507}{272}$
b. $\text{cov}(X, Y) = \frac{-169}{272}$,
c. $\mathbb{E}(U) = \frac{13}{2}$, $\text{var}(U) = \frac{169}{272}$

3.18. Let X, Y, and Z denote the number of spades, hearts, and diamonds respectively, in the hand.

a. $\mathbb{P}(X = i, Y = j | Z = 4) = \frac{\binom{13}{i}\binom{13}{j}\binom{22}{9-i-j}}{\binom{48}{9}}$ where i and j are nonnegative integers with $i + j \le 9$.
b. $\mathbb{P}(X = i | Y = 3, Z = 2) = \frac{\binom{13}{i}\binom{34}{8-i}}{\binom{47}{8}}$ for $i \in \{0, 1, ..., 8\}$.

4. Order Statistics

4.16.

a. $\mathbb{P}(X_{5,3} = k) = \frac{\binom{k-1}{2}\binom{25-k}{2}}{\binom{25}{5}}$ for $k \in \{3, 4, ..., 22\}$
b. $\mathbb{E}(X_{5,3}) = 13$
c. $\text{var}(X_{5,3}) = \frac{125}{7}$,

4.17. 1437
4.19. 2322

5. The Matching Problem

5.29.

a.

k	0	1	2	3	4	5
$b_5(k)$	44	45	20	10	0	1

b.

k	0	1	2	3	4	5
$\mathbb{P}(N_5 = k)$	0.3667	0.3750	0.1667	0.0833	0	0.0083

c. Covariance: $\frac{1}{100}$, correlation $\frac{1}{16}$

5.30.

a.

k	$\mathbb{P}(N_{10} = k)$
0	0.3678794
1	0.3678791
2	0.1839409

3	0.0613095
4	0.0153356
5	0.0030555
6	0.0005208
7	0.0000661
8	0.0000124
9	0
10	0.0000003

b. $\mathbb{E}(N_{10}) = 1$, $\text{var}(N_{10}) = 1$

c. $\mathbb{P}(N_{10} \geq 3) = 0.0803$

5.31.

a. See 5.30 (a)

c.

k	$\mathbb{P}(N = k)$
0	0.3678794
1	0.3678794
2	0.1839397
3	0.0613132
4	0.0153283
5	0.0030657
6	0.0005109
7	0.0000730
8	0.0000091
9	0.0000014
10	0.0000001

6. The Birthday Problem

6.7. $\frac{5}{18}$

6.9. $\frac{89}{144}$

6.11. $\frac{189}{625}$

6.13. $n = 9$

6.27.

a. $\mathbb{P}\left(V_{365,30} = j\right) = \binom{30}{j} \sum_{k=0}^{j} (-1)^k \binom{j}{k} \left(\frac{j-k}{365}\right)^{30}$ for $j \in \{1, 2, ..., 30\}$

b. $\mathbb{E}\left(V_{365,30}\right) = 28.8381$

c. $\text{var}\left(V_{365,30}\right) = 1.0458$

d. $\mathbb{P}(V_{365,30} \ge 28) = 0.89767$

6.29.

a. $\mathbb{P}(V_{6,10} = j) = \binom{10}{j} \sum_{k=0}^{j} (-1)^k \binom{j}{k} \left(\frac{j-k}{6}\right)^{10}$ for $j \in \{1, 2, ..., 6\}$
b. $\mathbb{E}(V_{6,10}) = 5.0310$
c. $\text{var}(V_{6,10}) = 0.5503$
d. $\mathbb{P}(V_{6,10} \le 4) = 0.22182$

6.31.

a. $\mathbb{P}(U_{10,15} = j) = \binom{15}{j} \sum_{k=0}^{10-j} (-1)^k \binom{10-j}{k} \left(1 - \frac{j+k}{10}\right)^{15}$ for $j \in \{0, 1, ..., 9\}$
b. $\mathbb{E}(V_{10,15}) = 2.0589$
c. $\text{var}(V_{10,15}) = 0.9864$
d. $\mathbb{P}(V_{10,15} \ge 3) = 0.3174$

6.33.

a.

j	1	2	3
$\mathbb{P}(V_{4,3} = j)$	$\frac{1}{16}$	$\frac{9}{16}$	$\frac{6}{16}$

b. $\mathbb{P}(V_{4,3} = 1) = \frac{1}{16}$
c. $\mathbb{E}(V_{4,3}) = \frac{37}{16}$
d. $\text{sd}(V_{4,3}) = 0.6830$

6.34.

a.

j	1	2	3	4	5
$\mathbb{P}(V_{10,5} = j)$	$\frac{1}{10000}$	$\frac{927}{2000}$	$\frac{9}{50}$	$\frac{63}{127}$	$\frac{189}{625}$

b. $\mathbb{E}(V_{10,5}) = 4.095$
c. $\text{sd}(V_{10,5}) = 0.727$

7. The Coupon Collector Problem

7.12. Let $W = W_{365,40}$ denote the sample size.

a. $\mathbb{P}(W = n) = \binom{364}{39} \sum_{j=0}^{39} (-1)^j \binom{39}{j} \left(\frac{39-j}{365}\right)^{n-1}$ for $n = \{40, 41, ...\}$
b. $\mathbb{E}(W) = 42.3049$
c. $\text{var}(W) = 2.4878$
d. $\mathbb{E}(t^W) = \prod_{i=1}^{40} \frac{366-i}{365-(i-1)\,t}$ for $|t| < \frac{365}{39}$

7.13. Let $W = W_{6,6}$ denote the number of throws.

a. $\mathbb{P}(W = n) = \sum_{j=0}^{5} (-1)^j \binom{5}{j} \left(\frac{5-j}{6}\right)^{n-1}$ for $n = \{6, 7, ...\}$
b. $\mathbb{E}(W) = 14.7$
c. $\text{var}(W) = 38.99$
d. $\mathbb{P}(W \geq 10) = 0.7436$

7.14. Let $W = W_{10,10}$ denote the number of boxes purchased.

a. $\mathbb{P}(W = n) = \sum_{j=0}^{9} (-1)^j \binom{9}{j} \left(\frac{9-j}{10}\right)^{n-1}$ for $n = \{10, 11, ...\}$
b. $\mathbb{E}(W) = 29.2897$
c. $\text{var}(W) = 125.6871$
d. $\mathbb{P}(W \leq 15) = 0.04595$