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## 6. Convergence

In this section we discuss several topics that are a bit advanced, but very important. In particular the results obtained in this section will be essential for establishing

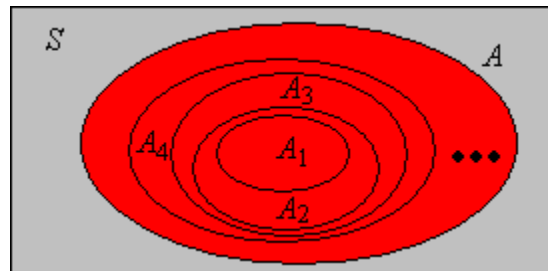
- [properties of distribution functions](#),
- [the weak law of large numbers](#),
- [the strong law of large numbers](#).

Some of the concepts from the section on [Partial Orders](#) in the chapter on [Foundations](#) are essential for this section. As usual, our starting point is a [random experiment](#) with [sample space](#)  $S$  and [probability measure](#)  $\mathbb{P}$ .

### The Continuity Theorems

#### Increasing Events

A sequence of events  $(A_1, A_2, \dots)$  is said to be **increasing** if  $A_n \subseteq A_{n+1}$  for each  $n$ . Thus, the events are increasing with respect to the subset partial order. The terminology is also justified by considering the corresponding indicator variables.



1. Let  $I_n$  denote the indicator variable of the event  $A_n$  for  $n \in \mathbb{N}_+$ . Show that the sequence of events is increasing if and only if the sequence of indicator variables is increasing in the ordinary sense. That is,  $I_n \leq I_{n+1}$  for each  $n$ .

If  $(A_1, A_2, \dots)$  is an increasing sequence of events, we refer to the union of the events as the **limit** of the events:

$$\lim_{n \rightarrow \infty} A_n = \bigcup_{n=1}^{\infty} A_n$$

Once again, the terminology is clarified by the corresponding indicator variables.

2. Suppose that  $(A_1, A_2, \dots)$  is an increasing sequence of events. Let  $I_n$  denote the indicator variable of  $A_n$  for  $n \in \mathbb{N}_+$ , and let  $I$  denote the indicator variable of the union of the events. Show that

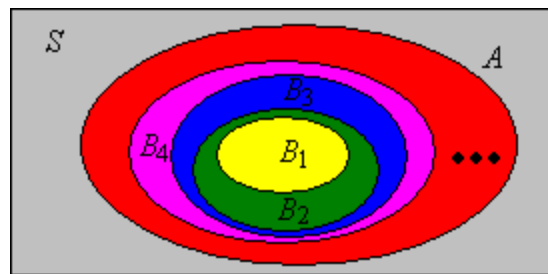
$$\lim_{n \rightarrow \infty} I_n = I$$

Generally speaking, a function is continuous if it preserves limits. Thus, the result in the following exercise is referred to as the **continuity theorem for increasing events**:

3. Suppose that  $(A_1, A_2, \dots)$  is an increasing sequence of events. Show that

$$\mathbb{P}\left(\lim_{n \rightarrow \infty} A_n\right) = \lim_{n \rightarrow \infty} \mathbb{P}(A_n)$$

- a. Let  $B_1 = A_1$  and let  $B_i = A_i \setminus A_{i-1}$  for  $i \in \{2, 3, \dots\}$ .
- b. Show that the collection of events  $\{B_1, B_2, \dots\}$  is pairwise disjoint and has the same union as  $\{A_1, A_2, \dots\}$
- c. Use the **additivity axiom of probability** and the definition of an infinite series.



An arbitrary union of events can always be written as a union of increasing events, as the next exercise shows.

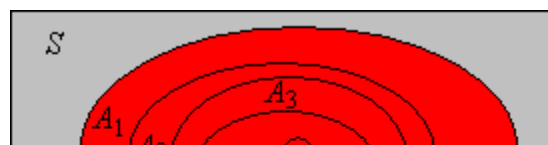
4. Suppose that  $(A_1, A_2, \dots)$  is a sequence of events. Show that

- a.  $\bigcup_{i=1}^n A_i$  is increasing in  $n$
- b.  $\lim_{n \rightarrow \infty} \bigcup_{i=1}^n A_i = \bigcup_{i=1}^{\infty} A_i$
- c.  $\lim_{n \rightarrow \infty} \mathbb{P}\left(\bigcup_{i=1}^n A_i\right) = \mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right)$

5. Suppose that  $A$  is an event for a basic experiment with  $\mathbb{P}(A) > 0$ . In the compound experiment that consists of independent replications of the basic experiment, show that the event “ $A$  eventually occurs” has probability 1.

### Decreasing Events

A sequence of events  $(A_1, A_2, \dots)$  is said to be **decreasing** if  $A_{n+1} \subseteq A_n$  for each  $n$ . Thus, the events are decreasing with respect to the subset partial order. The terminology is also justified by considering the corresponding indicator variables.





6. Let  $I_n$  denote the indicator variable of an event  $A_n$  for  $n \in \mathbb{N}_+$ . Show that the sequence of events is decreasing if and only if the sequence of indicator variables is decreasing in the ordinary sense. That is,  $I_{n+1} \leq I_n$  for each  $n$ .

If  $(A_1, A_2, \dots)$  is a decreasing sequence of events, we refer to the intersection of the events as the **limit** of the events:

$$\lim_{n \rightarrow \infty} A_n = \bigcap_{n=1}^{\infty} A_n$$

Once again, the terminology is clarified by the corresponding indicator variables.

7. Suppose that  $(A_1, A_2, \dots)$  is a decreasing sequence of events. Let  $I_n$  denote the indicator variable of  $A_n$  for  $n \in \mathbb{N}_+$ , and let  $I$  denote the indicator variable of the intersection of the events. Show that

$$\lim_{n \rightarrow \infty} I_n = I$$

The result in the following exercise is referred to as the **continuity theorem for decreasing events** :

8. Suppose that  $(A_1, A_2, \dots)$  is a decreasing sequence of events. Show that

$$\mathbb{P}\left(\lim_{n \rightarrow \infty} A_n\right) = \lim_{n \rightarrow \infty} \mathbb{P}(A_n)$$

- Show that the sequence of events  $(A_1^c, A_2^c, \dots)$  is increasing.
- Apply the continuity theorem for increasing events to the sequence in part (a).
- Use [DeMorgan's law](#) and the [complement rule](#).

An arbitrary intersection of events can always be written as an intersection of decreasing events, as the next exercise shows.

9. Suppose that  $(A_1, A_2, \dots)$  is a sequence of events. Show that

- $\bigcap_{i=1}^n A_i$  is decreasing in  $n$
- $\lim_{n \rightarrow \infty} \bigcap_{i=1}^n A_i = \bigcap_{i=1}^{\infty} A_i$
- $\lim_{n \rightarrow \infty} \mathbb{P}\left(\bigcap_{i=1}^n A_i\right) = \mathbb{P}\left(\bigcap_{i=1}^{\infty} A_i\right)$

## The Borel-Cantelli Lemmas

### The First Lemma

Suppose that  $(A_1, A_2, \dots)$  is an arbitrary sequence of events.

10. Show that  $\bigcup_{i=n}^{\infty} A_i$  is decreasing in  $n$ .

The limit (that is, the intersection) of the decreasing sequence in the previous exercise is called the **limit superior** of the original sequence.

$$\limsup_{n \rightarrow \infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} A_i$$

11. Show that  $\limsup_{n \rightarrow \infty} A_n$  is the event that occurs if and only if  $A_n$  occurs for infinitely many values of  $n$ .

Once again, the terminology is justified by the corresponding indicator variables:

12. Let  $I_n$  denote the indicator variable of  $A_n$  for  $n \in \mathbb{N}_+$ , and let  $I$  denote the indicator variable of  $\limsup_{n \rightarrow \infty} A_n$ . Show that

$$I = \limsup_{n \rightarrow \infty} I_n$$

13. Use the continuity theorem for decreasing events to show that

$$\mathbb{P}\left(\limsup_{n \rightarrow \infty} A_n\right) = \lim_{n \rightarrow \infty} \mathbb{P}\left(\bigcup_{i=n}^{\infty} A_i\right)$$

The result in the next exercise is the **first Borel-Cantelli Lemma**, named after **Emil Borel** and **Francesco Cantelli**. It gives a condition that is sufficient to conclude that infinitely many events occur with probability 0.

14. Show that if  $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$  then  $\mathbb{P}\left(\limsup_{n \rightarrow \infty} A_n\right) = 0$ .

- Start with the result of the previous exercise.
- Apply **Boole's inequality**.
- Recall that if an infinite series converges, then the tail of the series converges to 0.

## The Second Lemma

In this section we suppose that  $(A_1, A_2, \dots)$  is an arbitrary sequence of events.

15. Show that  $\bigcap_{i=n}^{\infty} A_i$  is increasing in  $n$ .

The limit (that is, the union) of the increasing sequence in the previous exercise is called the **limit inferior** of the original sequence.

$$\liminf_{n \rightarrow \infty} A_n = \bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} A_i$$

16. Show that  $\liminf_{n \rightarrow \infty} A_n$  is the event that occurs if and only if  $A_n$  occurs for all but finitely many values of  $n$ .

Once again, the terminology is justified by the corresponding indicator variables:

17. Let  $I_n$  denote the indicator variable of  $A_n$  for  $n \in \mathbb{N}_+$ , and let  $I$  denote the indicator variable of  $\liminf_{n \rightarrow \infty} A_n$ . Show that

$$I = \liminf_{n \rightarrow \infty} I_n$$

18. Use the continuity theorem for decreasing events to show that

$$\mathbb{P}\left(\liminf_{n \rightarrow \infty} A_n\right) = \lim_{n \rightarrow \infty} \mathbb{P}\left(\bigcap_{i=n}^{\infty} A_i\right)$$

19. Show that  $\liminf_{n \rightarrow \infty} A_n \subseteq \limsup_{n \rightarrow \infty} A_n$ .

20. Use DeMorgan's law to show that  $\left(\limsup_{n \rightarrow \infty} A_n\right)^c = \liminf_{n \rightarrow \infty} A_n^c$ .

The result in the next exercise is the **second Borel-Cantelli Lemma**. It gives a condition that is sufficient to conclude that infinitely many events occur with probability 1.

21. Suppose that  $(A_1, A_2, \dots)$  is a sequence of independent events. Show that if  $\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty$  then

$$\mathbb{P}\left(\limsup_{n \rightarrow \infty} A_n\right) = 1.$$

- Show that  $1 - x \leq e^{-x}$  for every  $x \in \mathbb{R}$
- Conclude that  $1 - \mathbb{P}(A_k) \leq e^{-\mathbb{P}(A_k)}$  for each  $k$ .
- Now use the result of the previous exercise and independence.

22. Suppose that  $A$  is an event in a basic experiment with  $\mathbb{P}(A) > 0$ . Show that in the compound experiment that consists of independent replications of the basic experiment, the event “ $A$  occurs infinitely often” has probability 1.

23. Suppose that we have an infinite sequence of coins labeled 1, 2, ... Moreover, coin  $n$  has probability of heads  $\frac{1}{n^a}$  for each  $n \in \mathbb{N}_+$ , where  $a > 0$  is a parameter. We toss each coin in sequence one time. In terms of  $a$ , find the probability of the following events:

- infinitely many heads.
- infinitely many tails.



## Convergence of Random Variables

Suppose that  $(X_1, X_2, \dots)$  and  $X$  are real-valued random variables for an experiment. We will discuss two ways that the sequence  $X_n$  can “converge” to  $X$  as  $n \rightarrow \infty$ . These are fundamentally important concepts, since some of the deepest results in probability theory are limit theorems.

First, we say that  $X_n \rightarrow X$  as  $n \rightarrow \infty$  **with probability 1** if

$$\mathbb{P}(X_n \rightarrow X \text{ as } n \rightarrow \infty) = 1$$

The statement that an event has probability 1 is the strongest statement that we can make in probability theory. Thus, convergence with probability 1 is the strongest form of convergence. The phrases **almost surely** and **almost everywhere** are sometimes used instead of the phrase *with probability 1*.

Next we say that  $X_n \rightarrow X$  as  $n \rightarrow \infty$  **in probability** if

$$\mathbb{P}(|X_n - X| > \varepsilon) \rightarrow 0 \text{ as } n \rightarrow \infty \text{ for each } \varepsilon > 0$$

The phrase *in probability* sounds superficially like the phrase *with probability 1*. However, as we will see, convergence in probability is much weaker than convergence with probability 1. Indeed, convergence with probability 1 is often called **strong convergence**, while convergence in probability is often called **weak convergence**. The next sequence of exercises explores convergence with probability 1. We will let  $\mathbb{Q}_+$  denote the set of positive rational numbers; a critical point to remember is that this set is countable.

24. Show that the following events are equivalent:

- $X_n$  does not converge to  $X$  as  $n \rightarrow \infty$ .
- For some  $\varepsilon > 0$ ,  $|X_n - X| > \varepsilon$  for infinitely many  $n$ .
- For some  $\varepsilon \in \mathbb{Q}_+$ ,  $|X_n - X| > \varepsilon$  for infinitely many  $n$ .

25. Use the result of the previous exercise to show that the following are equivalent

- $\mathbb{P}(X_n \rightarrow X \text{ as } n \rightarrow \infty) = 1$
- $\mathbb{P}(|X_n - X| > \varepsilon \text{ for infinitely many } n) = 0$  for every  $\varepsilon \in \mathbb{Q}_+$ . *Hint: Use [Boole's inequality](#).*
- $\mathbb{P}(|X_n - X| > \varepsilon \text{ for infinitely many } n) = 0$  for every  $\varepsilon > 0$ .
- $\mathbb{P}(|X_k - X| > \varepsilon \text{ for some } k \geq n) \rightarrow 0$  as  $n \rightarrow \infty$  for every  $\varepsilon > 0$ .

Part (b) of this exercise and the [first Borel-Cantelli Lemma](#) lead to a nice criterion for convergence with probability 1:

26. Show that if  $\sum_{n=1}^{\infty} \mathbb{P}(|X_n - X| > \varepsilon) < \infty$  for every  $\varepsilon > 0$  then  $X_n \rightarrow X$  as  $n \rightarrow \infty$  with probability 1.

Part (c) of [Exercise 25](#) leads to one of our main results: convergence with probability 1 implies convergence in probability.

▣ 27. Show that if  $X_n \rightarrow X$  as  $n \rightarrow \infty$  with probability 1 then  $X_n \rightarrow X$  as  $n \rightarrow \infty$  in probability.

The converse fails with a passion as the next exercise shows.

▣ 28. As in [Exercise 23](#), suppose that we have a sequence of coins labeled 1, 2, ...; coin  $n$  lands heads up with probability  $\frac{1}{n}$  for each  $n$ . We toss the coins in order to produce a sequence  $(X_1, X_2, \dots)$  of independent indicator random variables with

$$\mathbb{P}(X_n = 1) = \frac{1}{n}, \quad \mathbb{P}(X_n = 0) = 1 - \frac{1}{n}, \quad n \in \mathbb{N}_+$$

- Use the second Borel-Cantelli lemma to show that  $\mathbb{P}(X_n = 0 \text{ for infinitely many } n) = 1$ , so that infinitely many tails occur with probability 1.
- Use the second Borel-Cantelli lemma to show that  $\mathbb{P}(X_n = 1 \text{ for infinitely many } n) = 1$ , so that infinitely many heads occur with probability 1.
- Use (a) and (b) to show that  $\mathbb{P}(X_n \text{ does not converge as } n \rightarrow \infty) = 1$ .
- Show that  $X_n \rightarrow 0$  as  $n \rightarrow \infty$  in probability.

However, there is a partial converse to [Exercise 27](#) that is very useful.

▣ 29. Show that if  $X_n \rightarrow X$  as  $n \rightarrow \infty$  in probability, then there exists a subsequence  $(n_1, n_2, n_3, \dots)$  of  $\mathbb{N}_+$  such that  $X_{n_k} \rightarrow X$  as  $k \rightarrow \infty$  with probability 1.

- Assuming convergence in probability, show that for each  $k \in \mathbb{N}_+$  there exists  $n_k \in \mathbb{N}_+$  such that  $\mathbb{P}\left(|X_{n_k} - X| > \frac{1}{k}\right) < \frac{1}{k^2}$ .
- We can make the choices in part (a) so that  $n_k < n_{k+1}$  for each  $k$ .
- From (a) and (b) conclude that  $\sum_{k=1}^{\infty} \mathbb{P}\left(|X_{n_k} - X| > \varepsilon\right) < \infty$  for every  $\varepsilon > 0$ .
- Use [Exercise 26](#) to conclude that  $X_{n_k} \rightarrow X$  as  $k \rightarrow \infty$  with probability 1.
- The proof works because  $\frac{1}{k} \rightarrow 0$  as  $k \rightarrow \infty$  and  $\sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$ . Any two sequences with these properties would work just as well.

There are two other modes of convergence that we will discuss later:

- [convergence in mean](#),
- [convergence in distribution](#).