# Answers to Selected Exercises

## 2. Probability Spaces



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### 2. Events and Random Variables

2.18.

b. $Y(x_1, x_2, ..., x_n) = x_1 + x_2 + \cdots + x_n$. The set of possible values is $\{0, 1, ..., n\}$
d. $\{11100, 11010, 11001, 10110, 10101, 10011, 01110, 01101, 01011, 00111\}$

2.20.

b. $Y(x_1, x_2, ..., x_n) = x_1 + x_2 + \cdots + x_n$. The set of possible values is $\{n, n+1, ..., n\,k\}$
c. $U(x_1, x_2, ..., x_n) = \min\{x_1, x_2, ..., x_n\}$. The set of possible values is $\{1, 2, ..., k\}$
d. $V(x_1, x_2, ..., x_n) = \max\{x_1, x_2, ..., x_n\}$. The set of possible values is $\{1, 2, ..., k\}$
e. $\left\{(u, v) \in \{1, 2, ..., k\}^2 : u \le v\right\}$

2.21.

a. $A = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6)\}$
b. $B = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$
c. $A \cup B = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$
d. $A \cap B = \{(1, 6)\}$
e. $A^c \cap B^c = (A \cup B)^c = \{(2, 1), (2, 2), (2, 3), (2, 4), (2, 6), (3, 1), (3, 2), (3, 3), (3, 5), (3, 6), (4, 1), (4, 2), (4, 4), ($

2.23.

c. $\{X_1 < 3, X_2 > 4\} = \{(1, 5), (2, 5), (1, 6), (2, 6)\}$
d. $\{Y = 7\} = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (2, 5), (6, 1)\}$
e. $\{U = V\} = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)\}$

2.25. Let $D_5 = \{(1, 4), (2, 3), (3, 2), (4, 1)\}$, $D_7 = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$, $D = D_5 \cup D_7$, and $C = \{1, 2, 3, 4, 5, 6\}^2 \setminus D$.

a. $S = D \cup (C \times D) \cup \left(C^2 \times D\right) \cup \cdots$, $A = D_5 \cup (C \times D_5) \cup \left(C^2 \times D_5\right) \cup \cdots$
b. $S = D$, $A = D_5$

2.26.

a. $S = \{1, 2, 3, 4, 5, 6\}^3$
b. $W(x_1, x_2, x_3) = \mathbf{1}(x_1 = 6) + \mathbf{1}(x_2 = 6) + \mathbf{1}(x_3 = 6) - 1$

2.27. Let 1 denote heads and 0 tails for a coin toss.

a. $\bigcup_{n=1}^{6} \{0, 1\}^n$, $\#(S) = 126$
b. $N(x_1, x_2, ..., x_n) = n$ for $(x_1, x_2, ..., x_n) \in S$
c. $Y(x_1, x_2, ..., x_n) = \sum_{i=1}^{n} x_i$ for $(x_1, x_2, ..., x_n) \in S$
d. $\{Y = 2\} = \{11, 011, 101, 110, 0011, 0101, 0110, 1001, 1010, 1100, 00011, 00101, 00110, 01001, 01010, 011($

2.29. For the coin, let 1 denote heads and 0 tails.

a. $S = \{0, 1\} \times \{1, 2, 3, 4, 5, 6\}$, $\#(S) = 12$
b. $X(i, j) = i$ for $(i, j) \in S$
c. $Y(i, j) = j$ for $(i, j) \in S$
d. $\{Y \ge 4\} = \{0, 1\} \times \{4, 5, 6\}$

2.31.

c. 311875200, 2598960
d. 3954242643911239680000, 635013559600

2.32.

a. $Q = \{$q♣, q♦, q♥, q♠$\}$
b. $H = \{$1♥, 2♥, ..., 10♥, j♥, q♥, k♥$\}$
c. $Q \cup H = \{$1♥, 2♥, ..., 10♥, j♥, q♥, k♥, q♣, q♦, q♠$\}$
d. $Q \cap H = \{$q♥$\}$
e. $Q \setminus H = \{$q♣, q♦, q♠$\}$

2.34.

a. The set of possible values of $V$ is $\{0, 1, ..., 37\}$
b. $\#(\{V = 0\}) = 2310789600$

2.36.

a. $\#(A) = 3744$
b. $\#(B) = 624$
c. $\#(A) = 5148$

2.38.

a. $S = \left[-\frac{1}{2}, \frac{1}{2}\right]^2$
b. $A = \left[r - \frac{1}{2}, \frac{1}{2} - r\right]^2$
c. $A^c = \left\{(x, y) \in S : \left(x < r - \frac{1}{2} \text{ or } x > \frac{1}{2} - r \text{ or } y < r - \frac{1}{2} \text{ or } y > \frac{1}{2} - r\right)\right\}$

d. $Z(x, y) = \sqrt{x^2 + y^2}$ for $(x, y) \in S$
e. $\{X < Y\} = \{(x, y) \in S : x < y\}$
f. $\left\{Z < \frac{1}{2}\right\} = \left\{(x, y) \in S : x^2 + y^2 < \frac{1}{4}\right\}$

2.42.

a. 254251200
b. 2118760
c. 658008, 913900, 444600, 936000, 8400, 252

2.47.

a. $U_{3,1} = X_1 + X_2 + X_3 - X_1 X_2 - X_1 X_3 - X_2 X_3 + X_1 X_2 X_3$
b. $U_{3,2} = X_1 X_2 + X_1 X_3 + X_2 X_3 - 2 X_1 X_2 X_3$
c. $U_{3,3} = X_1 X_2 X_3$

2.48. $Y = X_3 (X_1 + X_2 - X_1 X_2) (X_4 + X_5 - X_4 X_5) + (1 - X_3) (X_1 X_4 + X_2 X_5 - X_1 X_2 X_4 X_5)$

2.55. For gender, let 0 denote female and 1 male. For species, let 1 denote tredecula, 2 tredecim, and 3 tredecassini.

a. $S = (0, \infty)^4 \times \{0, 1\} \times \{1, 2, 3\}$
b. $F = \{(x_1, x_2, x_3, x_4, y, z) \in S, y = 0\}$
e. $S^{104}$ where $S$ is given in (a).

2.56.

a. $S = \mathbb{N}^6 \times (0, \infty)$
b. $A = \{(n_1, n_2, n_3, n_4, n_5, n_6, w) \in S : n_1 + n_2 + n_3 + n_4 + n_5 + n_6 > 57\}$
e. $S^{30}$ where $S$ is given in (a).

## 3. Probability Measure

3.29.

a. $A$ occurs but not $B$. $\mathbb{P}(A \setminus B) = \frac{7}{30}$
b. $A$ or $B$ occurs. $\mathbb{P}(A \cup B) = \frac{29}{60}$
c. One of the events does not occur. $\mathbb{P}((A \cap B)^c) = \frac{9}{10}$
d. Neither event occurs. $\mathbb{P}((A \cup B)^c) = \frac{31}{60}$
e. Either $A$ occurs or $B$ does not occur. $\mathbb{P}(A \cup B^c) = \frac{17}{20}$

3.30.

a. $\mathbb{P}(A \cup B \cup C) = 0.67$
b. $\mathbb{P}((A \cup B \cup C)^c) = 0.33$
c. $\mathbb{P}((A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c) \cup (A^c \cap B^c \cap C)) = 0.45$
d. $\mathbb{P}((A \cap B \cap C^c) \cup (A \cap B^c \cap C) \cup (A^c \cap B \cap C)) = 0.21$

3.31.

a. $\mathbb{P}(A) = \frac{1}{4}$
b. $\mathbb{P}(B) = \frac{1}{3}$
c. $\mathbb{P}(A \cup B) = \frac{1}{2}$
d. $\mathbb{P}(A^c \cup B^c) = \frac{11}{12}$
e. $\mathbb{P}(A^c \cap B^c) = \frac{1}{2}$

3.32.

a. $\mathbb{P}(B) = \frac{1}{2}$
b. $\mathbb{P}(A \setminus B) = \frac{1}{5}$
c. $\mathbb{P}(B \setminus A) = \frac{3}{10}$
d. $\mathbb{P}(A^c \cup B^c) = \frac{4}{5}$
e. $\mathbb{P}(A^c \cap B^c) = \frac{3}{10}$

3.33.

d. Probabilities of $Y$

| $k$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $\mathbb{P}(Y = k)$ | $\frac{1}{32}$ | $\frac{5}{32}$ | $\frac{10}{32}$ | $\frac{10}{32}$ | $\frac{5}{32}$ | $\frac{1}{32}$ |

3.34.

a. $\mathbb{P}(A) = \frac{1}{2}$
b. $\mathbb{P}(B) = \frac{3}{8}$
c. $\mathbb{P}(A \cap B) = \frac{1}{4}$
d. $\mathbb{P}(A \cup B) = \frac{5}{8}$
e. $\mathbb{P}(A^c \cup B^c) = \frac{3}{4}$
f. $\mathbb{P}(A^c \cap B^c) = \frac{3}{8}$
g. $\mathbb{P}(A \cup B^c) = \frac{7}{8}$

3.37.

a. $A = \{X_1 < 3\}$
b. $B = \{X_1 + X_2 = 6\}$
c. $\mathbb{P}(A) = \frac{1}{3}$
d. $\mathbb{P}(B) = \frac{5}{36}$
e. $\mathbb{P}(A \cap B) = \frac{2}{36}$

f. $\mathbb{P}(A \cup B) = \frac{5}{12}$

g. $\mathbb{P}(B \setminus A) = \frac{1}{12}$

3.39.

a. $\mathbb{P}(Y = y) = \frac{6 - |y - 7|}{36}$ for $y \in \{2, 3, ..., 12\}$

b. $\mathbb{P}(U = u) = \frac{13 - 2\,u}{36}$ for $u \in \{1, 2, ..., 6\}$

c. $\mathbb{P}(V = v) = \frac{2\,v - 1}{36}$ for $v \in \{1, 2, ..., 6\}$

d. $\mathbb{P}(U = u, V = v) = \begin{cases} \frac{2}{36}, & u < v \\ \frac{1}{36}, & u = v \end{cases}$

3.40. Let $D_5 = \{(1, 4), (2, 3), (3, 2), (4, 1)\}$, $D_7 = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$, $D = D_5 \cup D_7$, and $C = \{1, 2, 2, 4, 5, 6\}^2 \setminus D$.

a. $S = D \cup (C \times D) \cup \left(C^2 \times D\right) \cup \cdots$

b. $A = D_5 \cup (C \times D_5) \cup \left(C^2 \times D_5\right) \cup \cdots$

c. $\mathbb{P}(A) = \frac{2}{5}$

d. $S = D$

e. $A = D_5$

f. $\mathbb{P}(A) = \frac{2}{5}$

3.42.

a. $\mathbb{P}(H_1) = \frac{1}{4}$

b. $\mathbb{P}(H_1 \cap H_2) = \frac{1}{17}$

c. $\mathbb{P}(H_2 \setminus H_1) = \frac{13}{68}$

d. $\mathbb{P}(H_2) = \frac{1}{4}$

e. $\mathbb{P}(H_1 \cup H_2) = \frac{15}{34}$

3.44.

a. $\frac{3744}{2598960} \approx 0.001441$

b. $\frac{624}{2598960} \approx 0.000240$

c. $\frac{5148}{2598960} \approx 0.001981$

3.46. $\frac{347373600}{635013559600} \approx 0.000547$

3.47.

a. $\frac{151519319380}{635013559600} \approx 0.2386$

b. $\frac{47079732700}{635013559600} \approx 0.0741$
c. $\frac{11404407300}{635013559600} \approx 0.0179$

3.48.

a. $\frac{1913496}{2598960} \approx 0.7363$
b. $\frac{32427298180}{635013559600} \approx 0.0511$

3.49.

a. $S = \left[-\frac{1}{2}, \frac{1}{2}\right]^2$
b. Since the coin is tossed "randomly," no region of $S$ should be preferred over any other.
c. $\left\{r - \frac{1}{2} < X < \frac{1}{2} - r,\ r - \frac{1}{2} < Y < \frac{1}{2} - r\right\}$
d. $\mathbb{P}(A) = (1 - 2\,r)^2$
e. $\mathbb{P}(A^c) = 1 - (1 - 2\,r)^2$
f. $\mathbb{P}\left(Z < \frac{1}{2}\right) = \frac{\pi}{4}$

3.53.

Probabilities of $Y$

| $k$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $\mathbb{P}(Y = k)$ | $\frac{2584}{23751}$ | $\frac{8075}{23751}$ | $\frac{950}{2639}$ | $\frac{3800}{23751}$ | $\frac{100}{3393}$ | $\frac{2}{1131}$ |

3.55. Let $U$ denote the urn (as a set of 12 balls).

a. $S$ is the set of all subsets (combinations) of size 3 chosen from $U$.
b. $\mathbb{P}(A) = \frac{3}{44}$
c. $\mathbb{P}(B) = \frac{3}{11}$

3.56. Again let $U$ denote the urn (as a set of 12 balls).

a. $S = U^3$
b. $\mathbb{P}(A) = \frac{1}{8}$
c. $\mathbb{P}(B) = \frac{5}{24}$

3.57

a. $\mathbb{P}(A) = 1, \mathbb{P}(B) = 0, \mathbb{P}(C) = 0$
b. $\mathbb{P}(A) = 0, \mathbb{P}(B) = 0, \mathbb{P}(C) = 1$
c. $\mathbb{P}(A) = \frac{1}{4}, \mathbb{P}(B) = \frac{1}{2}, \mathbb{P}(C) = \frac{1}{4}$
d. $\mathbb{P}(A) = 0, \mathbb{P}(B) = 1, \mathbb{P}(C) = 0$
e. $\mathbb{P}(A) = \frac{1}{2}, \mathbb{P}(B) = \frac{1}{2}, \mathbb{P}(C) = 0$
f. $\mathbb{P}(A) = 0, \mathbb{P}(B) = \frac{1}{2}, \mathbb{P}(C) = \frac{1}{2}$

3.58

a. $\mathbb{P}(B) = 0$, $\mathbb{P}(C) = 0$, $\mathbb{P}(D) = 0$
b. $\mathbb{P}(B) = \frac{1}{2}$, $\mathbb{P}(C) = 0$, $\mathbb{P}(D) = \frac{1}{2}$
c. $\mathbb{P}(B) = 0$, $\mathbb{P}(C) = \frac{1}{2}$, $\mathbb{P}(D) = 0$
d. $\mathbb{P}(B) = \frac{1}{2}$, $\mathbb{P}(C) = \frac{1}{2}$, $\mathbb{P}(D) = \frac{1}{2}$
e. $\mathbb{P}(B) = 1$, $\mathbb{P}(C) = \frac{1}{2}$, $\mathbb{P}(D) = 0$
f. $\mathbb{P}(B) = 1$, $\mathbb{P}(C) = 0$, $\mathbb{P}(D) = 1$

3.59

b. $e^{-3}$
c. $e^{-2} - e^{-4}$

3.60

b. $1 - \frac{5}{2} e^{-1}$
c. $\frac{17}{24} e^{-1}$

3.63

a. 0.6333333333
b. 0.6321205357
c. 0.6321205588

3.64.

a. $\mathbb{P}(R) = \frac{13}{30}$
b. $\mathbb{P}(T) = \frac{19}{30}$
c. $\mathbb{P}(W) = \frac{9}{30}$
d. $\mathbb{P}(R \cap T) = \frac{9}{30}$
e. $\mathbb{P}(T \setminus W) = \frac{11}{30}$

3.65.

a. $\mathbb{P}(W) = \frac{37}{104}$
b. $\mathbb{P}(F) = \frac{59}{104}$
c. $\mathbb{P}(T) = \frac{44}{104}$
d. $\mathbb{P}(W \cap F) = \frac{34}{104}$
e. $\mathbb{P}(W \cup T \cup F) = \frac{85}{104}$

## 4. Conditional Probability

4.7.

a. $\mathbb{P}(A|B) = \frac{2}{5}$
b. $\mathbb{P}(B|A) = \frac{3}{10}$
c. $\mathbb{P}(A^c|B) = \frac{3}{5}$
d. $\mathbb{P}(B^c|A) = \frac{7}{10}$
e. $\mathbb{P}(A^c|B^c) = \frac{31}{45}$

4.8.

a. $\mathbb{P}(A \cap B^c|C) = \frac{1}{4}$
b. $\mathbb{P}(A \cup B|C) = \frac{7}{12}$
c. $\mathbb{P}(A^c \cap B^c|C) = \frac{5}{12}$

4.9.

a. $\mathbb{P}(A \cap B) = \frac{1}{4}$
b. $\mathbb{P}(A \cup B) = \frac{7}{12}$
c. $\mathbb{P}(B \cup A^c) = \frac{3}{4}$
d. $\mathbb{P}(B|A) = \frac{1}{2}$
e. $A$ and $B$ are positively correlated.

4.10. For a person chosen at random from the population, let $S$ denote the event that the person smokes and $D$ the event that the person has the disease.

a. $\mathbb{P}(D \cap S) = 0.036$
b. $\mathbb{P}(S|D) = 0.45$
c. $S$ and $D$ are positively correlated.

4.11.

a. $\mathbb{P}(X > 30) = \frac{2}{3}$
b. $\mathbb{P}(X > 45|X > 30) = \frac{1}{2}$
c. Given $X > 30$, $X$ is uniformly distributed on $(30, 60)$

4.12.

a. $\mathbb{P}(X_1 = 3) = \frac{1}{6}$, $\mathbb{P}(Y = 5) = \frac{1}{9}$, $\mathbb{P}(X_1 = 3|Y = 5) = \frac{1}{4}$, $\mathbb{P}(Y = 5|X_1 = 3) = \frac{1}{6}$. The events are positively correlated.
b. $\mathbb{P}(X_1 = 3) = \frac{1}{6}$, $\mathbb{P}(Y = 7) = \frac{1}{6}$, $\mathbb{P}(X_1 = 3|Y = 7) = \frac{1}{6}$, $\mathbb{P}(Y = 7|X_1 = 3) = \frac{1}{6}$. The events are independent.
c. $\mathbb{P}(X_1 = 2) = \frac{1}{6}$, $\mathbb{P}(Y = 5) = \frac{1}{9}$, $\mathbb{P}(X_1 = 2|Y = 5) = \frac{1}{4}$, $\mathbb{P}(Y = 5|X_1 = 2) = \frac{1}{6}$. The events are positively correlated.
d. $\mathbb{P}(X_1 = 3) = \frac{1}{6}$, $\mathbb{P}(X_1 = 2) = \frac{1}{6}$, $\mathbb{P}(X_1 = 3|X_1 = 2) = 0$, $\mathbb{P}(X_1 = 2|X_1 = 3) = 0$. The events are negatively correlated.

4.14. The conditional distribution of $(X_1, X_2)$ given $Y = 7$ is uniform on $\{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$

4.15. Let $X$ denote the die score and $H$ the event that all coin tosses result in heads.

a. $\mathbb{P}(H) = \frac{21}{128}$
b. $\mathbb{P}(X = i | H) = \frac{64}{63} \frac{1}{2^i}, \quad i \in \{1, 2, 3, 4, 5, 6\}$

4.17. Let $V$ denote the probability of heads for the randomly selected coin, and $H$ the event that the coin lands heads.

a. $\mathbb{P}(H) = \frac{41}{72}$
b. $\mathbb{P}(V = p | H) = \begin{cases} \frac{15}{41}, & p = \frac{1}{2} \\ \frac{8}{41}, & p = \frac{1}{3} \\ \frac{18}{41}, & p = 1 \end{cases}$

4.18. Let $X$ denote the die score and $H$ the even that the coin lands heads.

a. $\mathbb{P}(X = i) = \begin{cases} \frac{5}{24}, & i \in \{1, 6\} \\ \frac{7}{48}, & i \in \{2, 3, 4, 5\} \end{cases}$
b. $\mathbb{P}(H | X = 4) = \frac{3}{7}$, $\mathbb{P}(H^c | X = 4) = \frac{4}{7}$

4.20.

a. $\mathbb{P}(Q_1) = \frac{1}{13}$, $\mathbb{P}(H_1) = \frac{1}{4}$, $\mathbb{P}(Q_1 | H_1) = \frac{1}{13}$, $\mathbb{P}(H_1 | Q_1) = \frac{1}{4}$, independent.
b. $\mathbb{P}(Q_1) = \frac{1}{13}$, $\mathbb{P}(Q_2) = \frac{1}{13}$, $\mathbb{P}(Q_1 | Q_2) = \frac{3}{51}$, $\mathbb{P}(Q_2 | Q_1) = \frac{3}{51}$, negatively correlated.
c. $\mathbb{P}(Q_2) = \frac{1}{13}$, $\mathbb{P}(H_2) = \frac{1}{4}$, $\mathbb{P}(Q_2 | H_2) = \frac{1}{13}$, $\mathbb{P}(H_2 | Q_2) = \frac{1}{4}$, independent..
d. $\mathbb{P}(Q_1) = \frac{1}{13}$, $\mathbb{P}(H_2) = \frac{1}{4}$, $\mathbb{P}(Q_1 | H_2) = \frac{1}{13}$, $\mathbb{P}(H_2 | Q_1) = \frac{1}{4}$, independent.

4.22. Let $H_i$ denote the event that card $i$ is a heart and $S_i$ the event that card $i$ is a spade.

a. $\mathbb{P}(H_1 \cap H_2 \cap H_3) = \frac{11}{850}$
b. $\mathbb{P}(H_1 \cap H_2 \cap S_3) = \frac{13}{850}$
c. $\mathbb{P}(H_1 \cap S_2 \cap H_3) = \frac{13}{850}$

4.24.

a. $\mathbb{P}(X > 0 | X < Y) = \frac{3}{4}$
b. Given $(X, Y) \in \left[r - \frac{1}{2}, \frac{1}{2} - r\right]^2$, $(X, Y)$ is uniformly distributed on this set.

4.26. Let $R$ denote the number of reds and $W$ the weight. $\mathbb{P}(R \geq 10 | W \geq 48) = \frac{10}{23}$

4.27. Let $M$ denote the event that a cicada is male, $U$ the event that the cicada is tredula, and $W$ the body weight.

a. $\mathbb{P}(W \ge 0.25 | M) = \frac{2}{45}$
b. $\mathbb{P}(W \ge 0.25 | U) = \frac{7}{44}$

4.28. Let $X$ denote the production line of the selected item, and $D$ the event that the item is defective.

a. $\mathbb{P}(D) = 0.037$
b. $\mathbb{P}(X = i | D) = \begin{cases} 0.541, & i = 1 \\ 0.405, & i = 2 \\ 0.054, & i = 3 \end{cases}$

4.29.

a. $\frac{7}{8}$
b. $\frac{1}{7}$

4.30.

a. $\frac{23}{24}$
b. $\frac{3}{23}$

4.31.

a. 5.55% of the population is colorblind.
b. 90.9% of colorblind persons are male.

4.32.

a. $\frac{5}{6}$
b. $\frac{1}{6}$
c. $\frac{1}{5}$

4.33. Let $G$ denote the event that the ball is green and $U_1$ the event that urn 1 is chosen.

a. $\mathbb{P}(G) = \frac{9}{20}$
b. $\mathbb{P}(U_1 | G) = \frac{2}{3}$

4.34. Let $G_i$ denote the event that the ball from urn $i$ is green.

a. $\mathbb{P}(G_2) = \frac{9}{25}$
b. $\mathbb{P}(G_1 | G_2) = \frac{2}{3}$

4.35. Let $R_i$ denote the event that the ball $i$ is red and $G_i$ the event that ball $i$ is green.

c. $\mathbb{P}(R_1 \cap R_2 \cap G_3) = \frac{a\, b\, (a+k)}{(a+b)\, (a+b+k)\, (a+b+2\, k)}$

d. $\mathbb{P}(R_1 \cap G_2 \cap R_3) = \frac{a\,b\,(a+k)}{(a+b)\,(a+b+k)\,(a+b+2\,k)}$
e. $\mathbb{P}(G_1 \cap R_2 \cap R_3) = \frac{a\,b\,(a+k)}{(a+b)\,(a+b+k)\,(a+b+2\,k)}$
f. $\mathbb{P}(R_2) = \frac{a}{a+b}$
g. $\mathbb{P}(R_1 | R_2) = \frac{a+k}{a+b+k}$

4.36. Let $R_i$ denote the event that the ball $i$ is red and $G_i$ the event that ball $i$ is green.

a. $\mathbb{P}(R_1 \cap R_2 \cap G_3) = \frac{9}{55}$
b. $\mathbb{P}(R_1 \cap G_2 \cap R_3) = \frac{7}{44}$
c. $\mathbb{P}(G_1 \cap R_2 \cap R_3) = \frac{49}{330}$
d. $\mathbb{P}(R_2) = \frac{32}{55}$
e. $\mathbb{P}(R_1 | R_2) = \frac{9}{16}$

4.39. 0.905.
4.40. 0.949.
4.41. 0.268.
4.42. 0.9098.

## 5. Independence

5.8

a. $A$, $B$, $C$ are independent if and only if
   1. $\mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B)$
   2. $\mathbb{P}(A \cap C) = \mathbb{P}(A)\,\mathbb{P}(C)$
   3. $\mathbb{P}(B \cap C) = \mathbb{P}(B)\,\mathbb{P}(C)$
   4. $\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A)\,\mathbb{P}(B)\,\mathbb{P}(C)$
b. $A$, $B$, $C$, $D$ are independent if and only if
   1. $\mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B)$
   2. $\mathbb{P}(A \cap C) = \mathbb{P}(A)\,\mathbb{P}(C)$
   3. $\mathbb{P}(A \cap D) = \mathbb{P}(A)\,\mathbb{P}(D)$
   4. $\mathbb{P}(B \cap C) = \mathbb{P}(B)\,\mathbb{P}(C)$
   5. $\mathbb{P}(B \cap D) = \mathbb{P}(B)\,\mathbb{P}(D)$
   6. $\mathbb{P}(C \cap D) = \mathbb{P}(C)\,\mathbb{P}(D)$
   7. $\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A)\,\mathbb{P}(B)\,\mathbb{P}(C)$
   8. $\mathbb{P}(A \cap B \cap D) = \mathbb{P}(A)\,\mathbb{P}(B)\,\mathbb{P}(D)$
   9. $\mathbb{P}(A \cap C \cap D) = \mathbb{P}(A)\,\mathbb{P}(C)\,\mathbb{P}(D)$
   10. $\mathbb{P}(B \cap C \cap D) = \mathbb{P}(B)\,\mathbb{P}(C)\,\mathbb{P}(D)$
   11. $\mathbb{P}(A \cap B \cap C \cap D) = \mathbb{P}(A)\,\mathbb{P}(B)\,\mathbb{P}(C)\,\mathbb{P}(D)$

5.20.

a. $\mathbb{P}(A \cap B \cap C) = 0.12$

b. $\mathbb{P}(A^c \cap B^c \cap C^c) = 0.07$
c. $\mathbb{P}(A \cup B \cup C) = 0.93$
d. $\mathbb{P}((A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c) \cup (A^c \cap B^c \cap C)) = 0.38$
e. $\mathbb{P}((A \cap B \cap C^c) \cup (A \cap B^c \cap C) \cup (A^c \cap B \cap C)) = 0.43$

5.21.

a. $\mathbb{P}((A \cap B) \cup C) = \frac{3}{8}$
b. $\mathbb{P}(A \cup B^c \cup C) = \frac{7}{8}$
c. $\mathbb{P}((A^c \cap B^c) \cup C^c) = \frac{5}{6}$

5.22. There should be 9 women executives.

5.23. The probability that the students select the same tire is $\frac{1}{16}$. .

5.26.

a. $\mathbb{P}(Q_1) = \mathbb{P}(Q_1 | H_1) = \frac{1}{13}$, $\mathbb{P}(H_1) = \mathbb{P}(H_1 | Q_1) = \frac{1}{4}$,
b. $\mathbb{P}(Q_2) = \mathbb{P}(Q_2 | H_2) = \frac{1}{13}$, $\mathbb{P}(H_2) = \mathbb{P}(H_2 | Q_2) = \frac{1}{4}$,
c. $\mathbb{P}(Q_1) = \mathbb{P}(Q_2) = \frac{1}{13}$, $\mathbb{P}(Q_2 | Q_1) = \mathbb{P}(Q_1 | Q_2) = \frac{1}{17}$.
d. $\mathbb{P}(H_1) = \mathbb{P}(H_2) = \frac{1}{4}$, $\mathbb{P}(H_2 | H_1) = \mathbb{P}(H_1 | H_2) = \frac{4}{17}$,
e. $\mathbb{P}(Q_1) = \mathbb{P}(Q_1 | H_2) = \frac{1}{13}$, $\mathbb{P}(H_2) = \mathbb{P}(H_2 | Q_1) = \frac{1}{4}$,
f. $\mathbb{P}(Q_2) = \mathbb{P}(Q_2 | H_1) = \frac{1}{13}$, $\mathbb{P}(H_1) = \mathbb{P}(H_1 | Q_2) = \frac{1}{4}$,

5.28. Let $A$ denote the event of at least one six. $\mathbb{P}(A) = 1 - \left(\frac{5}{6}\right)^5 \approx 0.5981$

5.29. Let $A$ denote the event of at least one double six. $\mathbb{P}(A) = 1 - \left(\frac{35}{36}\right)^{10} \approx 0.2455$

5.31. Let $F$ denote the event that a sum of 4 occurs before a sum of 7. $\mathbb{P}(F) = \frac{1}{3}$

5.32.

$$\mathbb{P}(Y = k) = \begin{cases} \frac{32}{243}, & k = 0 \\ \frac{80}{243}, & k = 1 \\ \frac{80}{243}, & k = 2 \\ \frac{40}{243}, & k = 3 \\ \frac{10}{243}, & k = 4 \\ \frac{1}{243}, & k = 5 \end{cases}$$

5.37.

a. $\mathbb{P}(X < Y) = \frac{11}{12}$

b. $\mathbb{P}(X > 20,\ Y > 20) = \frac{8}{27}$

5.42.

a. $R = 0.504$
b. $R = 0.902$
c. $R = 0.994$

5.43. The 5-engine plane would be preferable if $p > \frac{1}{2}$ (which one would hope would be the case). The 3-engine plane would be preferable if $p < \frac{1}{2}$. If $p = \frac{1}{2}$, the 3-engine and 5-engine planes are equally reliable.

5.44. Consider cases, depending on whether component 3 is working or failed:

a. $Y(X_1, X_2, X_3, X_4, X_5) = X_3\,(X_1 + X_2 - X_1\,X_2)\,(X_4 + X_5 - X_4\,X_5) + (1 - X_3)\,(X_1\,X_4 + X_2\,X_5 - X_1\,X_2\,X_4\,X_5)$
b. $R(p_1, p_2, p_3, p_4, p_5) = p_3\,(p_1 + p_2 - p_1\,p_2)\,(p_4 + p_5 - p_4\,p_5) + (1 - p_3)\,(p_1\,p_4 + p_2\,p_5 - p_1\,p_2\,p_4\,p_5)$

5.45. Let $L$ denote the event that the conditions are low stress and $W$ the event that the system works

a. $\mathbb{P}(W) = 0.9917$
b. $\mathbb{P}(L|W) = 0.504$

5.48. Let $A$ denote the event that the woman is pregnant and $T_i$ the event that test $i$ is positive. $\mathbb{P}(A|T_1 \cap {T_2}^c \cap T_3) = 0.834$

5.49.

a. sensitivity $1 - (1 - a)^3$, specificity $b^3$.
b. sensitivity $3\,a^2\,(1 - a) + a^3$, specificity $b^3 + 3\,b^2\,(1 - b)$.
c. sensitivity $a^3$, specificity $1 - (1 - b)^3$.

5.50. Let $C$ denote the event that the defendant is convicted and $G$ the event that the defendant is guilty.

a. $\mathbb{P}(C) = 0.51458$
b. $\mathbb{P}(G|C) = 0.99996$
c. The independence assumption is ridiculous, of course, since jurors collaborate.

5.51.

a. $\frac{987}{1024}$
b. $\frac{27}{987}$

5.52. Let $C$ denote the event that the mother is a carrier and let $S_i$ denote the event that son $i$ is healthy.

a. $\mathbb{P}(S_1 \cap S_2) = \frac{5}{8}$
b. $\mathbb{P}(C|S_1 \cap S_2) = \frac{1}{5}$
c. $\mathbb{P}(S_3|S_1 \cap S_2) = \frac{9}{10}$

5.57. $\frac{11}{12}$. No, not really.

## 6. Convergence

6.23. Let $H_n$ be the event that toss $n$ results in heads, and $T_n$ the event that toss $n$ results in tails.

a. $\mathbb{P}\left(\limsup_{n \to \infty} H_n\right) = 1$, $\mathbb{P}\left(\limsup_{n \to \infty} T_n\right) = 1$ if $a \in (0, 1]$
b. $\mathbb{P}\left(\limsup_{n \to \infty} H_n\right) = 0$, $\mathbb{P}\left(\limsup_{n \to \infty} T_n\right) = 1$ if $a \in (1, \infty)$

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