5. Splitting a Poisson Process

The Two-Type Process

Suppose that $\{N_t : t \ge 0\}$ is a Poisson process with rate $r > 0$. Suppose additionally that each arrival, independently of the others, is of one of two types: type 1 with probability p and type 0 with probability $q = 1 - p$. This is sometimes referred to as **splitting** a Poisson process. Here are some common examples:

- The arrivals are radioactive emissions and that each emitted particle is either *detected* (type 1) or *missed* (type 0) by a counter.
- The arrivals are customers at a service station and each customer is classified as either *male* (type 1) or *female* (type 0).

The Joint Distribution

We are interested in the type 1 arrivals and the type 0 arrivals jointly. For $t \ge 0$, let M_t denote the number of type 1 arrivals in the interval $(0, t]$ and let $W_t = N_t - M_t$ denote the number of type 0 arrivals in $(0, t]$

1. Use the definition of conditional probability, to show that for $t \ge 0$,

$$\mathbb{P}(M_t = j, W_t = k) = \mathbb{P}(M_t = j | N_t = j + k)\, \mathbb{P}(N_t = j + k), \quad (j, k) \in \mathbb{N}^2$$

2. Argue that in terms of *type*, the successive arrivals form a Bernoulli trials process, and hence if there are $j + k$ arrivals in the interval $(0, t]$, then the number of type 1 arrivals has the binomial distribution with trial parameter $j + k$ and success parameter p.

3. Use the results of Exercises 1 and 2 to show that

$$\mathbb{P}(M_t = j, W_t = k) = e^{-r\,p\,t} \frac{(r\,p\,t)^j}{j!} e^{-r\,(1-p)\,t} \frac{(r\,(1-p)\,t)^k}{k!}, \quad (j, k) \in \mathbb{N}^2$$

From Exercise 3, it follows that the number of type 1 arrivals in the interval $(0, t]$ and the number of type 2 arrivals in the interval $(0, t]$ are independent, and have Poisson distributions with parameters $r\,p\,t$ and $r\,(1 - p)\,t$, respectively. More generally, $\{M_t : t \ge 0\}$ and $\{W_t : t \ge 0\}$ form separate, independent Poisson processes with rates $r\,p$ and $r\,(1 - p)$, respectively.

4. In the two-type Poisson experiment vary r, p, and t with the scroll bars and note the shape of the probability density functions. Now set $r = 2$, $t = 3$, $p = 0.7$. Run the experiment 1000 times with an update frequency of 10 and watch the apparent convergence of the relative frequency functions to the density functions.

5. In the two-type Poisson experiment, set $r = 2$, $t = 3$, $p = 0.7$. Run the experiment 500 times, updating

after each run. Compute the appropriate relative frequency functions and investigate empirically the independence of the number of men and the number of women.

6. Suppose that customers arrive at a service station according to the Poisson model, with rate $r = 20$ per hour. Moreover, each customer, independently, is female with probability 0.6 and male with probability 0.4. Find the probability that in a 2 hour period, there will be at least 20 women and at least 15 men.

Estimating the Number of Arrivals

Suppose that the type 1 arrivals are observable, but not the type 0 arrivals. This setting is natural, for example, if the arrivals are radioactive emissions, and the type 1 arrivals are emissions that are detected by a counter, while the type 0 arrivals are emissions that are missed. Suppose that for a given $t > 0$, we would like to estimate the total number arrivals N_t after observing the number of type 1 arrivals M_t.

7. Show that the conditional distribution of N_t given $M_t = k$ is the same as the distribution of $k + W_t$.

8. Show that $\mathbb{E}(N_t | M_t = k) = k + r(1 - p)t$.

Thus, if the overall rate r and the probability p that an arrival is type 1 are known, then it follows form the general theory of conditional expectation that the best estimator of N_t based on M_t, in the least squares sense, is

$$\mathbb{E}(N_t | M_t) = M_t + r(1 - p)t$$

9. Show that $\mathbb{E}\left((N_t - \mathbb{E}(N_t | M_t))^2\right) = r(1 - p)t$

10. In the two-type Poisson experiment, set $r = 3$, $t = 4$, and $p = 0.8$. Run the experiment 100 times, updating after each run.

a. Compute the estimate of N_t based on M_t for each run.
b. Over the 100 runs, compute average of the sum of the squares of the errors.
c. Compare the result in (b) with the result in Exercise 9.

11. Suppose that a piece of radioactive material emits particles according to the Poisson model at a rate of $r = 100$ per second. Moreover, assume that a counter detects each emitted particle, independently, with probability 0.9. Suppose that the number of detected particles in a 5 second period is 465.

a. Estimate the number of particles emitted.
b. Compute the mean square error of the estimate.

The Multi-Type Process

Suppose that each arrival in the Poisson process, independently, is of one of k-types: type i with probability

p_i for $i \in \{1, 2, ..., k\}$. Of course we must have

$$\sum_{i=1}^{k} p_i = 1$$

Let $M_{i,t}$ denote the number of type i arrivals in the interval $(0, t]$ for $t \geq 0$ and $i \in \{1, 2, ..., k\}$.

12. Show that for fixed $t \geq 0$, $(M_{1,t}, M_{2,t}, ..., M_{k,t})$ is a sequence of independent random variables and $M_{i,t}$ has the Poisson distribution with parameter $r\, p_i\, t$.

More generally, $\{M_{i,t} : t \geq 0\}$ is a Poisson process with rate $r\, p_i$ for $i \in \{1, 2, ..., k\}$, and these Poisson processes are independent.
