1. The Fire Process

Modeling the Fire Process

In this section, we will study the spread of fire through a forest. As you will see, we will make many simplifying assumptions, yet even so, the resulting random process, called the **fire process**, is extremely complicated. It is an example of an **interacting particle system** (sometimes also called a **probabilistic cellular automaton**). In general, interacting particle systems are spatial configurations of particles (trees in this case) whose states change probabilistically, with the state of a given particle influenced by the states of its neighbors. The particles typically have very simple *local* interactions, yet *globally* the behavior of the particle system is very complex. Because of the complexity, the interest is usually in the asymptotic (long-term) behavior of the process.

We will consider an ideal forest that consists of a rectangular lattice of trees. That is, there is a tree at each point $(i,\ j)$ of the lattice. Each tree (except those on the boundary of the lattice) has four neighbors. The neighbors of $(i,\ j)$ are $\{(i+1,\ j),\ (i-j,\ j),\ (i,\ j+1),\ (i,\ j-1)\}$

At any time, each tree will be in one of three basic states: **healthy**, **on fire**, or **burnt**. Time is discrete, and the dynamics of the process are as follows:

- A tree that is on fire at time t will be burnt at time $t+1$.
- Once a tree is burnt it remains burnt.
- If a tree is healthy at time t and is directly above or below or to the right or to the left of a tree on fire at time t, then the healthy tree will catch on fire at time, $t+1$ independently, with respective probabilities $(p_u,\ p_d,\ p_r,\ p_l)$.
- Trees that are healthy at time t catch on fire at time $t+1$ independently of one another.

1. Show that, for example, if the healthy tree is above and to the right of trees that are on fire at time t (but the other two neighboring trees are healthy), then the healthy tree will catch on fire at time $t+1$ with probability $p_u + p_r - p_u\, p_r$

The directional probabilities may be used to model directional effects such as wind or terrain.

2. The main simplifying assumptions are a perfect lattice of trees, discrete time, and fire spread only through neighboring trees. Discuss the validity of these assumptions for a real forest fire.

3. In the fire experiment, select the 100 by 50 forest and set a single tree on fire in the center. Run the simulation and note whether or not the fire burns out, the general shape of the burn region, and the number and size of the islands of healthy trees. Repeat with various fire-spread probabilities. Can you draw any general conclusions?

The Isotropic Forest

Suppose now that we have an infinite forest with a single type of healthy tree for which the directional probabilities are the same, $p_u = p_d = p_r = p_l$, and let p denote the common value. We will call this as an **isotropic forest**. Some theoretical results are known for the isotropic forest:

1. The **critical value** of p is $\frac{1}{2}$. That is, starting with finitely many trees on fire, the fire will eventually die out with probability 1 if $p < \frac{1}{2}$. On the other hand, starting with at least one tree on fire, there is a positive probability that the fire will burn forever if $p > \frac{1}{2}$.
2. Moreover, when the fire burns forever, starting with a single tree on fire, the set of burnt trees will have an asymptotic shape: ball shaped for p near $\frac{1}{2}$ and diamond shaped for p near 1.

The fact that the asymptotic shape is diamond for large p is due to the neighborhood structure of the lattice (think about what happens when $p = 1$.

4. In the fire experiment, select the 500 by 250 forest and set a single tree on fire in the center. Run the simulation with constant fire-spread probability $p = 0.45$ until the fire either burns out or reaches the boundary of the forest. Repeat with $p = 0.5$, $p = 0.6$, $p = 0.7$, $p = 0.8$, and $p = 0.9$. In each case, note the frequency and size of the islands of green trees. Note the asymptotic shape of the burn region. Plot the number of trees on fire as a function of t.

Critical behavior and asymptotic shape results are typical for interacting particle systems.

One-Dimensional Fire Model

5. In the fire experiment, select the 100 by 50 forest. Now set $p_u = p_d = 0$ and set a single tree on fire. Run simulations with different values of the left and right fire-spread probabilities. Can you formulate any general conclusions? Note that you essentially have a one-dimensional forest.

Now consider an infinite, one-dimensional forest with a single type of healthy tree and with a single tree on fire initially. Let L denote the number of trees to the left of the initial tree that will be burnt and R the number of trees to the right of the initial tree that will be burnt (the initial tree is included in these counts).

6. Show that the random variables R and L are independent, and have geometric distributions with parameters $1 - p_r$ and $1 - p_l$, respectively.

By Exercise 6, if $p_l < 1$, then

$$\mathbb{P}(L = k) = (1 - p_l)\, {p_l}^{k-1}, \quad k \in \mathbb{N}_+$$

and in particular, $\mathbb{P}(L < \infty) = 1$. On the other hand, if $p_l = 1$, then $\mathbb{P}(L = \infty) = 1$.

Similarly, if $p_r < 1$, then

$$\mathbb{P}(R = k) = (1 - p_r)\, p_r^{\,k-1}, \quad k \in \mathbb{N}_+$$

and in particular, $\mathbb{P}(R < \infty) = 1$. On the other hand, if $p_r = 1$, then $\mathbb{P}(R = \infty) = 1$.

Thus, we have results for the one-dimensional forest that are analogous to those for the two-dimensional forest: The critical value for each parameter is 1, and the shape of the burn region is always an interval.

Additional Fire Experiments

7. Consider a forest with $p_d = p_l = 0$ and $p_u = p_r = p$. In the fire experiment, select the 500 by 250 forest with a single tree on fire in the center. Run the simulation with various values of p, and try to determine experimentally the approximate critical value of p. What can you say about the asymptotic shape?

8. Consider a forest tree with $p_d = 0$ and $p_u = p_l = p_r = p$. In the fire experiment, select the 500 by 250 forest with a single tree on fire in the center. Run the simulation with various values of p, and try to determine experimentally the approximate critical value of p. What can you say about the asymptotic shape?

9. Consider a forest with $p_l = 0$, $p_r = 1$, $p_u = 0$, and $p_d = r$. Thus, the fire is guaranteed to burn to the right and may burn downward; the fire will not burn to the left or upward. In the fire experiment, select the 500 by 250 forest with a single tree on fire in the upper left corner. Run the simulation a few times and try to describe the upper envelope of the burn region in terms of the Bernoulli trials process.