8. Reliability Chains

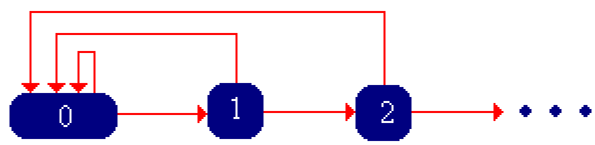
The Success-Runs Chain

Suppose that we have a sequence of **trials**, each of which results in either **success** or **failure**. Our basic assumption is that if there have been x consecutive successes, then the probability of success on the next trial is $p(x)$ where $p : \mathbb{N} \to (0, 1)$. Whenever there is a failure, we start over, independently, with a new sequence of trials. Appropriately enough, p is called the **success function**. Let X_n denote the length of the run of successes after n trials.

1. Argue that $\boldsymbol{X} = (X_0, X_1, X_2, ...)$ is a Markov chain with state space $\mathbb{N}$ and transition probability function

$$P(x, x + 1) = p(x), \;\; P(x, 0) = 1 - p(x), \quad x \in \mathbb{N}$$

This Markov chain is called the **success-runs chain**. The state graph of is given below:



Now let T denote the trial number of the first failure, starting with a fresh sequence of trials. Note that in the context of the success-runs chain $\boldsymbol{X}$, $T = T_0$, the first return time to state 0, starting in 0. Note that T takes values in $\mathbb{N}_+ \cup \{\infty\}$, since, presumably, it is possible that no failure occurs. Let $r(n) = \mathbb{P}(T > n)$ for $n \in \mathbb{N}$, the probability of at least n consecutive successes, starting with a fresh set of trials. Let $f(n) = \mathbb{P}(T = n)$ for $n \in \mathbb{N}_+$, the probability of exactly $n - 1$, consecutive successes, starting with a fresh set of trails.

2. Show that

a. $p(x) = \frac{r(x+1)}{r(x)}$ for $x \in \mathbb{N}$
b. $r(n) = \prod_{x=0}^{n-1} p(x)$ for $n \in \mathbb{N}$
c. $f(n) = (1 - p(n - 1)) \prod_{x=0}^{n-2} p(x)$ for $n \in \mathbb{N}_+$
d. $r(n) = 1 - \sum_{x=1}^{n} f(x)$ for $n \in \mathbb{N}$
e. $f(n) = r(n - 1) - r(n)$ for $n \in \mathbb{N}_+$

Thus, the functions p, r, and f give equivalent information. If we know one of the functions, we can construct the other two, and hence any of the functions can be used to define the success-runs chain.

3. The function r is the reliability function associated with T. Show that it is characterized by the following properties:

a. r is positive.
b. $r(0) = 1$
c. r is strictly decreasing.

4. Show that the function f is characterized by the following properties:

a. f is positive.
b. $\sum_{x=1}^{\infty} f(x) \le 1$

Essentially, f is the probability density function of T, except that it may be **defective** in the sense that the sum of its values may be less than 1. The leftover probability, of course, is the probability that $T = \infty$. This is the critical consideration in the classification of the success-runs chain, which we consider in the next paragraph.

5. Verify that each of the following functions has the appropriate properties, and then find the other two functions:

a. p is a constant in $(0, 1)$. Thus, the trials are Bernoulli trials.
b. $r(n) = \frac{1}{n+1}$ for $n \in \mathbb{N}$.
c. $r(n) = \frac{n+1}{2n+1}$ for $n \in \mathbb{N}$.
d. $p(x) = \frac{1}{x+2}$ for $x \in \mathbb{N}$.

6. The success-runs applet is a simulation of the success-runs chain based on Bernoulli trials. Run the simulation 1000 times for various values of p, and note the limiting behavior of the chain.

Recurrence and the Remaining Life Chain

7. From the state graph, show that the success-runs chain is irreducible and aperiodic.

Recall that T has the same distribution as the first return time to 0 starting at state 0. Thus, the classification of the chain as recurrent or transient depends on $\alpha = \mathbb{P}(T = \infty)$. Specifically, the success-runs chain is transient if $\alpha > 0$ and recurrent if $\alpha = 0$. Thus, we see that the chain is recurrent if and only if a failure is sure to occur. We can compute the parameter α in terms of each of the three functions that define the chain.

8. Show that

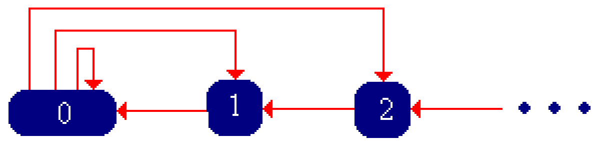
$$\alpha = \prod_{x=0}^{\infty} p(x) = \lim_{n \to \infty} r(n) = 1 - \sum_{x=1}^{\infty} f(x)$$

When the success-runs chain $\boldsymbol{X}$ is recurrent, we can define a related random process. Let Y_n denote the number of trials remaining until the next failure, after n trials.

9. Argue that $\boldsymbol{Y} = (Y_0, Y_1, Y_2, ...)$ is a Markov chain with state space $\mathbb{N}$ and transition probability function

$$Q(0, x) = f(x+1), \; Q(x+1, x) = 1, \quad x \in \mathbb{N}$$

10. The Markov chain $\boldsymbol{Y}$ is called the **remaining life** chain. Verify the state graph below and show that this chain is also irreducible, aperiodic, and recurrent.



11. Compute α and determine whether the success-runs chain $\boldsymbol{X}$ is transient or recurrent for each of the cases in Exercise 5.

12. Run the simulation of the success-runs chain 1000 times for various values of p, starting in state 0. Note the return times to state 0.

Positive Recurrence and Limiting Distributions

Let $\mu = \mathbb{E}(T)$, the expected trial number of the first failure, starting with a fresh sequence of trials.

13. Show that the success-runs chain $\boldsymbol{X}$ is positive recurrent if and only if $\mu < \infty$, in which case the invariant distribution has probability density function g given by

$$g(x) = \frac{r(x)}{\mu}, \quad x \in \mathbb{N}$$

14. Show that

a. If $\alpha > 0$ then $\mu = \infty$
b. If $\alpha = 0$ then $\mu = \sum_{n=1}^{\infty} n\, f(n)$
c. $\mu = \sum_{n=0}^{\infty} r(n)$

15. Suppose that $\alpha = 0$, so that the remaining life chain $\boldsymbol{Y}$ is well-defined. Show that this chain is also positive recurrent if and only if $\mu < \infty$, with the same invariant distribution as $\boldsymbol{X}$ (with probability density function g given in the previous exercise).

16. Determine whether the success-runs chain $\boldsymbol{X}$ is transient, null recurrent, or positive recurrent for each of the cases in Exercise 5. If the chain is positive recurrent, find the invariant probability density function.

17. The success-runs chain corresponding to Bernoulli trials has a geometric distribution as the invariant distribution. Run the simulation of the success-runs chain 1000 times for various values of p. Note the apparent convergence of the empirical distribution to the invariant distribution.

Time Reversal

Suppose that $\mu < \infty$, so that the success-runs chain $\boldsymbol{X}$ and the remaining-life chain $\boldsymbol{Y}$ are positive recurrent.

18. Show that $\boldsymbol{X}$ and $\boldsymbol{Y}$ are time reversals of each other, and use this fact to show again that g is the invariant probability density function.

19. Run the simulation of the success-runs chain 1000 times for various values of p, starting in state 0. If you imagine watching the simulation backwards in time, then you can see a simulation of the remaining life chain.
