Answers to Selected Exercises

16. Markov Chains

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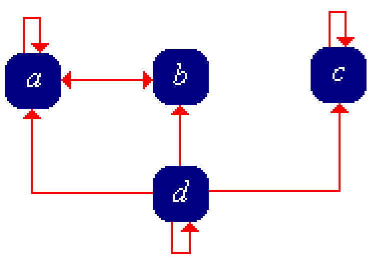
2. Transience and Recurrence

2.28.

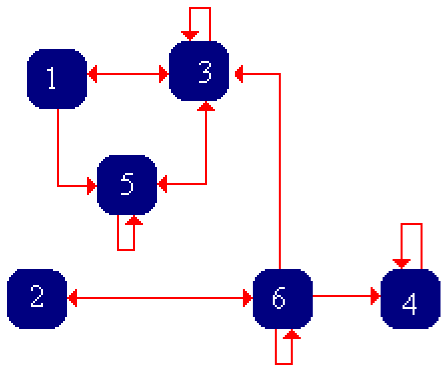
a. 

b. $\{a, b\}$ recurrent; $\{c\}$ recurrent; $\{d\}$ transient.

c. $G = \begin{pmatrix} \infty & \infty & 0 & 0 \\ \infty & \infty & 0 & 0 \\ 0 & 0 & \infty & 0 \\ \infty & \infty & \infty & \frac{1}{3} \end{pmatrix}$

d. $H = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} & \frac{1}{4} \end{pmatrix}$

2.29.

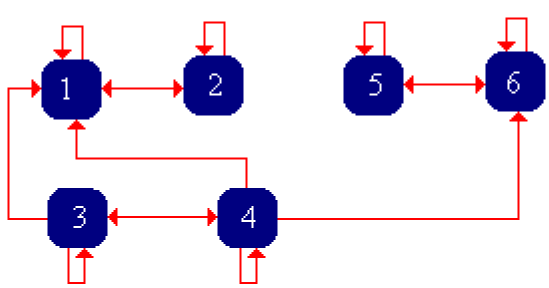
a. 

b. {1, 3, 5} recurrent; {2, 6} transient; {4} recurrent.

c. $G = \begin{pmatrix} \infty & 0 & \infty & 0 & \infty & 0 \\ \infty & \frac{1}{2} & \infty & \infty & \infty & 2 \\ \infty & 0 & \infty & 0 & \infty & 0 \\ 0 & 0 & 0 & \infty & 0 & 0 \\ \infty & 0 & \infty & 0 & \infty & 0 \\ \infty & \frac{1}{2} & \infty & \infty & \infty & 1 \end{pmatrix}$

d. $H = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

2.30.

a. 

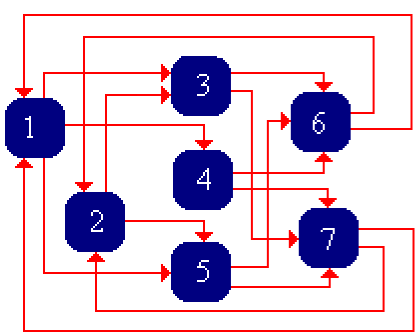
b. {1, 2} recurrent; {3, 4} transient; {5, 6} recurrent.

c. $G = \begin{pmatrix} \infty & \infty & 0 & 0 & 0 & 0 \\ \infty & \infty & 0 & 0 & 0 & 0 \\ \infty & \infty & \frac{7}{5} & \frac{4}{5} & \infty & \infty \\ \infty & \infty & \frac{4}{5} & \frac{3}{5} & \infty & \infty \\ 0 & 0 & 0 & 0 & \infty & \infty \\ 0 & 0 & 0 & 0 & \infty & \infty \end{pmatrix}$

d. $H = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ \frac{4}{5} & \frac{4}{5} & \frac{7}{12} & \frac{1}{2} & \frac{1}{5} & \frac{1}{5} \\ \frac{3}{5} & \frac{3}{5} & \frac{1}{3} & \frac{3}{8} & \frac{2}{5} & \frac{2}{5} \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$

3. Periodicity

3.5.

a. 

b. Period 3

c. $P^3 = \begin{pmatrix} \frac{71}{192} & \frac{121}{192} & 0 & 0 & 0 & 0 & 0 \\ \frac{29}{72} & \frac{43}{72} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{7}{18} & \frac{1}{12} & \frac{19}{36} & 0 & 0 \\ 0 & 0 & \frac{19}{48} & \frac{3}{32} & \frac{49}{96} & 0 & 0 \\ 0 & 0 & \frac{13}{32} & \frac{7}{64} & \frac{31}{64} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{157}{288} & \frac{131}{288} \\ 0 & 0 & 0 & 0 & 0 & \frac{37}{64} & \frac{27}{64} \end{pmatrix}$

d. Cyclic classes: $\{1, 2\}, \{3, 4, 5\}, \{6, 7\}$.

4. Invariant and Limiting Distributions

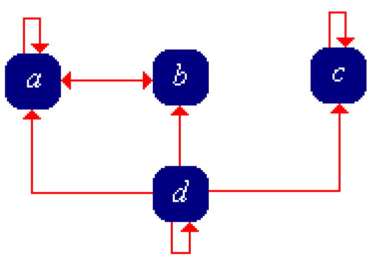
4.17.

a. $f = \left(\frac{q}{p+q}, \frac{p}{p+q}\right)$

b. $\mu = \left(\frac{p+q}{q}, \frac{p+q}{p}\right)$

c. $P^n \to \frac{1}{p+q}\begin{pmatrix} q & p \\ q & p \end{pmatrix}$ as $n \to \infty$.

4.18.

a. 

b. $\{a, b\}$ recurrent; $\{c\}$ recurrent; $\{d\}$ transient.

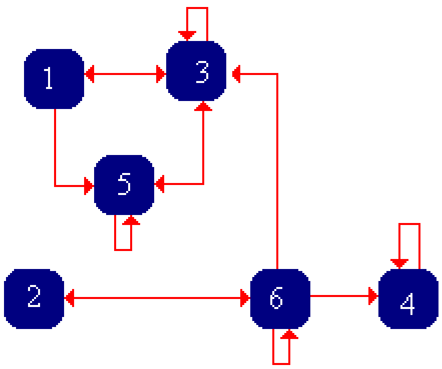
c. $f = \left(\frac{3}{5}p, \frac{2}{5}p, 1 - p, 0\right), 0 \le p \le 1$

d. $\mu = \left(\frac{5}{3}, \frac{5}{2}, 1, \infty\right)$

e. $P^n \to \begin{pmatrix} \frac{3}{5} & \frac{2}{5} & 0 & 0 \\ \frac{3}{5} & \frac{2}{5} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{2}{5} & \frac{4}{15} & \frac{1}{3} & 0 \end{pmatrix}$ as $n \to \infty$.

4.19.

a.



b. $\{1, 3, 5\}$ recurrent; $\{2, 6\}$ transient; $\{4\}$ recurrent.

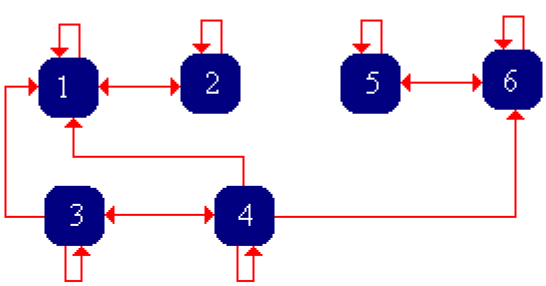
c. $f = \left(\frac{2}{19}p, 0, \frac{9}{19}p, 1 - p, \frac{9}{19}p, 0\right), 0 \le p \le 1$

d. $\mu = \left(\frac{19}{2}, \infty, \frac{19}{8}, 1, \frac{19}{9}, \infty\right)$

e. $P^n \to \begin{pmatrix} \frac{2}{19} & 0 & \frac{8}{19} & 0 & \frac{9}{19} & 0 \\ \frac{1}{19} & 0 & \frac{4}{19} & \frac{1}{2} & \frac{9}{38} & 0 \\ \frac{2}{19} & 0 & \frac{8}{19} & 0 & \frac{9}{19} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{2}{19} & 0 & \frac{8}{19} & 0 & \frac{9}{19} & 0 \\ \frac{1}{19} & 0 & \frac{4}{19} & \frac{1}{2} & \frac{9}{38} & 0 \end{pmatrix}$ as $n \to \infty$.

4.20.

a.



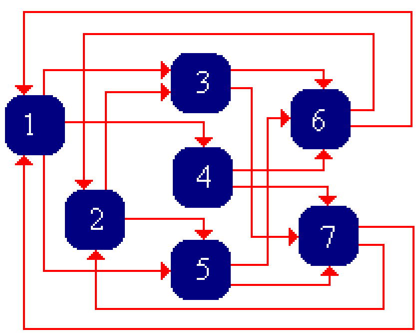
b. $\{1, 2\}$ recurrent; $\{3, 4\}$ transient; $\{5, 6\}$ recurrent.

c. $f = \left(\frac{1}{3}p, \frac{2}{3}p, 0, 0, \frac{1}{2}(1-p), \frac{1}{2}(1-p)\right), 0 \le p \le 1$

d. $\mu = \left(3, \frac{3}{2}, \infty, \infty, 2, 2\right)$

e. $P^n \to \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 & 0 \\ \frac{4}{15} & \frac{8}{15} & 0 & 0 & \frac{1}{10} & \frac{1}{10} \\ \frac{1}{5} & \frac{2}{5} & 0 & 0 & \frac{1}{5} & \frac{1}{5} \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ as $n \to \infty$.

4.21.

a. 

b. Cyclic classes: $\{1, 2\}, \{3, 4, 5\}, \{6, 7\}$.

c. $f = \frac{1}{1785}(232, 363, 237, 58, 300, 333, 262)$

d. $\mu = 1785\left(\frac{1}{232}, \frac{1}{363}, \frac{1}{237}, \frac{1}{58}, \frac{1}{300}, \frac{1}{333}, \frac{1}{262}\right)$

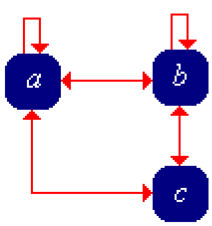
e. $P^{3n} \to \frac{1}{585}\begin{pmatrix} 232 & 363 & 0 & 0 & 0 & 0 & 0 \\ 232 & 363 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 237 & 58 & 300 & 0 & 0 \\ 0 & 0 & 237 & 58 & 300 & 0 & 0 \\ 0 & 0 & 237 & 58 & 300 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 333 & 262 \\ 0 & 0 & 0 & 0 & 0 & 333 & 262 \end{pmatrix}$ as $n \to \infty$.

f. $P^{3n+1} \to \frac{1}{585}\begin{pmatrix} 0 & 0 & 237 & 58 & 300 & 0 & 0 \\ 0 & 0 & 237 & 58 & 300 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 333 & 262 \\ 0 & 0 & 0 & 0 & 0 & 333 & 262 \\ 0 & 0 & 0 & 0 & 0 & 333 & 262 \\ 232 & 363 & 0 & 0 & 0 & 0 & 0 \\ 232 & 363 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ as $n \to \infty$.

g. $P^{3n+2} \to \frac{1}{585}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 333 & 262 \\ 0 & 0 & 0 & 0 & 0 & 333 & 262 \\ 232 & 363 & 0 & 0 & 0 & 0 & 0 \\ 232 & 363 & 0 & 0 & 0 & 0 & 0 \\ 232 & 363 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 237 & 58 & 300 & 0 & 0 \\ 0 & 0 & 237 & 58 & 300 & 0 & 0 \end{pmatrix}$ as $n \to \infty$.

5. Time Reversal

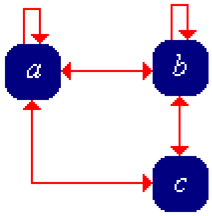
5.15.

a. 

b. $f = \left(\frac{6}{17}, \frac{6}{17}, \frac{5}{17}\right)$

c. $\mu = \left(\frac{17}{6}, \frac{17}{6}, \frac{17}{5}\right)$

d. $Q = \begin{pmatrix} \frac{1}{4} & \frac{1}{3} & \frac{5}{12} \\ \frac{1}{4} & \frac{1}{3} & \frac{5}{12} \\ \frac{3}{5} & \frac{2}{5} & 0 \end{pmatrix}$

e. 

6. The Ehrenfest Chains

6.2.

a. $f_1 = \left(\frac{1}{30}, \frac{7}{30}, \frac{7}{30}, \frac{7}{30}, \frac{7}{30}, \frac{1}{30}\right)$, $\mu_1 = \frac{5}{2}$, ${\sigma_1}^2 = \frac{19}{12}$

b. $f_2 = \left(\frac{7}{150}, \frac{19}{150}, \frac{49}{150}, \frac{49}{150}, \frac{19}{150}, \frac{7}{150}\right)$, $\mu_2 = \frac{5}{2}$, ${\sigma_2}^2 = \frac{79}{60}$

c. $f_3 = \left(\frac{19}{750}, \frac{133}{750}, \frac{223}{750}, \frac{223}{750}, \frac{133}{750}, \frac{19}{750}\right)$, $\mu_3 = \frac{5}{2}$, ${\sigma_3}^2 = \frac{431}{300}$

6.5.

a. $f_1 = (0, 0.2, 0.5, 0.3, 0, 0)$, $\mu_1 = 2.1$, $\sigma_1 = 0.7$

b. $f_2 = (0.02, 0.20, 0.42, 0.30, 0.06, 0)$, $\mu_2 = 2.18$, $\sigma_2 = 0.887$

c. $f_3 = (0.030, 0.194, 0.380, 0.300, 0.090, 0.006)$, $\mu_3 = 2.244$, $\sigma_3 = 0.984$

6.8. Returns to a state can only occur at even times. The cyclic classes are the set of even states and the set of odd states.

$$P^2(x, x-2) = \frac{x(x-1)}{m^2}, P^2(x, x) = \frac{x(m-x+1)+(m-x)(x+1)}{m^2}, P^2(x, x+2) = \frac{(m-x)(m-x-1)}{m^2}$$

6.11. $\mu(x) = \frac{2^m}{\binom{m}{x}}$.

6.12.

a. $P^{2n}(x, y) \to \binom{m}{y}\left(\frac{1}{2}\right)^{m-1}$ as $n \to \infty$ if $x \in S$, $y \in S$ have the same parity (both even or both odd). The limit is 0 otherwise.

b. $P^{2n+1}(x, y) \to \binom{m}{y}\left(\frac{1}{2}\right)^{m-1}$ as $n \to \infty$ if $x \in S$, $y \in S$ have opposite parity (one even and one odd). The limit is 0 otherwise.

6.13. $Q^n(x, y) \to \binom{m}{y}\left(\frac{1}{2}\right)^m$ as $n \to \infty$ for $x \in S$, $y \in S$

7. The Bernoulli-Laplace Chain

7.3.

a. $P(x, x-1) = \frac{(k-r+x)\,x}{k^2}$, $P(x, x) = \frac{(r-x)\,x+(k-r+x)\,(k-x)}{k^2}$, $P(x, x+1) = \frac{(r-x)\,(k-x)}{k^2}$,

$x \in \{\max\{0, r-k\}, ..., \min\{k, r\}\}$

b. $P(x, x-1) = \frac{x^2}{k^2}$, $P(x, x) = \frac{2\,x\,(k-x)}{k^2}$, $P(x, x+1) = \frac{(k-x)^2}{k^2}$, $x \in \{0, 1, ..., k\}$

7.4.

a. $P = \frac{1}{50}\begin{pmatrix} 30 & 20 & 0 & 0 & 0 \\ 7 & 31 & 12 & 0 & 0 \\ 0 & 16 & 28 & 6 & 0 \\ 0 & 0 & 27 & 21 & 2 \\ 0 & 0 & 0 & 40 & 10 \end{pmatrix}$

b. $f_1 = \frac{1}{250}(37, 67, 67, 67, 12)$, $\mu_1 = \frac{9}{5}$, ${\sigma_1}^2 = \frac{32}{25}$,

c. $f_2 = \frac{1}{12500}(1579, 3889, 4489, 2289, 254)$, $\mu_2 = \frac{83}{50}$, ${\sigma_2}^2 = \frac{2413}{2500}$,

d. $f_3 = \frac{1}{625000}(74593, 223963, 234163, 85163, 7118)$, $\mu_3 = \frac{781}{500}$, ${\sigma_3}^2 = \frac{206427}{250000}$,

7.8. $\mu(x) = \dfrac{\binom{j+k}{k}}{\binom{r}{x}\binom{j+k-r}{k-x}}, \quad x \in S$

7.9. $P^n(x, y) \to \dfrac{\binom{r}{y}\binom{j+k-r}{k-y}}{\binom{j+k}{k}}$ as $n \to \infty$ for $x \in S$, $y \in S$

8. Reliability Chains

8.5.

a. $p(x) = p,\ x \in \mathbb{N}$. $r(n) = p^n,\ n \in \mathbb{N}$. $f(n) = (1-p)\,p^{n-1},\ n \in \mathbb{N}_+$.

b. $p(x) = \frac{x+1}{x+2},\ x \in \mathbb{N}$. $r(n) = \frac{1}{n+1},\ n \in \mathbb{N}$. $f(n) = \frac{1}{n} - \frac{1}{n+1},\ n \in \mathbb{N}_+$.

c. $p(x) = \frac{(x+2)\,(2\,x+1)}{(x+1)\,(2\,x+3)},\ x \in \mathbb{N}$. $r(n) = \frac{n+1}{2\,n+1},\ n \in \mathbb{N}$. $f(n) = \frac{n}{2\,n-1} - \frac{n+1}{2\,n+1},\ n \in \mathbb{N}_+$.

d. $p(x) = \frac{1}{x+2},\ x \in \mathbb{N}$. $r(n) = \frac{1}{(n+1)!},\ n \in \mathbb{N}$. $f(n) = \frac{1}{n!} - \frac{1}{(n+1)!},\ n \in \mathbb{N}_+$.

8.11.

a. $\alpha = 0$, recurrent.
b. $\alpha = 0$, recurrent.
c. $\alpha = \frac{1}{2}$, transient.
d. $\alpha = 0$, recurrent.

8.16.

a. $\mu = \frac{1}{1-p}$, positive recurrent. $g(x) = (1-p)\,p^x$, $x \in \mathbb{N}$.

b. $\alpha = 0$, $\mu = \infty$, null recurrent.
c. $\alpha = \frac{1}{2}$, transient.
d. $\mu = e - 1$, positive recurrent. $g(x) = \frac{1}{(e-1)\,(x+1)!}$, $x \in \mathbb{N}$.

9. The Branching Chain

9.2.

a. $P(x, y) = e^{-m\,x} \frac{(m\,x)^y}{y!}$; $x \in \mathbb{N}, y \in \mathbb{N}$
b. $P(x, y) = \binom{k\,x}{y} p^y (1-p)^{k\,x-y}$; $x \in \mathbb{N}, y \in \{0, 1, ..., k\,x\}$
c. $P(x, y) = \binom{x+y-1}{x-1} p^y (1-p)^x$; $x \in \mathbb{N}, y \in \mathbb{N}$

9.3. Let $g(x, y, z) = P_1(X_1 = x, X_2 = y, X_3 = z)$.

a. $g(0, 0, 0) = 1 - p$
b. $g(2, 0, 0) = p\,(1-p)^2$
c. $g(2, 2, 0) = 2\,p^2\,(1-p)^3$
d. $g(2, 2, 2) = 4\,p^3\,(1-p)^2$
e. $g(2, 2, 4) = 2\,p^4\,(1-p)$
f. $g(2, 4, 0) = p^3\,(1-p)^4$
g. $g(2, 4, 2) = 4\,p^4\,(1-p)^3$
h. $g(2, 4, 4) = 6\,p^5\,(1-p)^2$
i. $g(2, 4, 6) = 4\,p^6\,(1-p)$
j. $g(2, 4, 8) = p^7$

9.13.

a. Mean $m = \frac{p}{1-p}$. Extinction probability $q = 1$ if $p \le \frac{1}{2}$; $q = \frac{1-p}{p}$ if $p \ge \frac{1}{2}$. Thus, $q = \frac{1}{m}$ when $m > 1$
b. Mean $m = 2\,p$. Extinction probability $q = 1$ if $p \le \frac{1}{2}$; $q = \frac{1-p}{p}$ if $p \ge \frac{1}{2}$. Curiously, the extinction probability is the same as for the offspring distribution in part (a).

10. The Queuing Chain

10.2.

a. $P(0, y) = e^{-m} \frac{m^y}{y!}$, $y \in \mathbb{N}$; $P(x, y) = e^{-m} \frac{m^{y-x+1}}{(y-x+1)!}$, $x \in \mathbb{N}_+$, $y \in \{x-1, x, x+1, ...\}$
b. $P(0, y) = \binom{k}{y} p^y (1-p)^{k-y}$, $y \in \mathbb{N}$; $P(x, y) = \binom{k}{y-x+1} p^{y-x} (1-p)^{k-y+x-1}$, $x \in \mathbb{N}_+$, $y \in \{x-1, x, ..., k+x-1\}$

c. $P(0, y) = (1 - p) p^y$, $y \in \mathbb{N}$; $P(x, y) = (1 - p) p^{y - x + 1}$, $x \in \mathbb{N}_+$, $y \in \{x - 1, x, x + 1, ...\}$

10.3. Let $g(x, y, z) = P_1(X_1 = x, X_2 = y, X_3 = z)$.

a. $g(0, 0, 0) = (1 - p)^3$
b. $g(0, 0, 2) = g(0, 2, 1) = g(2, 1, 0) = p (1 - p)^2$
c. $g(0, 2, 3) = g(2, 1, 3) = g(2, 3, 2) = p^2 (1 - p)$
d. $g(2, 3, 4) = p^3$

10.8.

a. $\Phi(t) = \frac{1 - p}{1 - p t}$. Hence $q = 1$ if $p \le \frac{1}{2}$ (so the chain is recurrent), and $q = \frac{1 - p}{p}$ if $p > \frac{1}{2}$ (so the chain is transient).
b. $\Phi(t) = (1 - p) + p t^2$. Hence $q = 1$ if $p \le \frac{1}{2}$ (so the chain is recurrent), and $q = \frac{1 - p}{p}$ if $p > \frac{1}{2}$ (so the chain is transient).

10.13.

a. $m = \frac{p}{1 - p}$. The chain is positive recurrent if $p < \frac{1}{2}$, null recurrent if $p = \frac{1}{2}$, and transient if $p > \frac{1}{2}$.
b. $m = 2 p$. The chain is positive recurrent if $p < \frac{1}{2}$, null recurrent if $p = \frac{1}{2}$, and transient if $p > \frac{1}{2}$.

12. Random Walks on Graphs

12.2. For the matrix and vector below, we use the ordered state space is $S = (a, b, c, d)$.

a. $$P = \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{2} \\ 0 & \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{5} & \frac{2}{5} & \frac{2}{5} & 0 \end{pmatrix}$$
b. $f_2 = \left(\frac{9}{40}, \frac{1}{5}, \frac{13}{40}, \frac{1}{4}\right)$

12.3. For the matrix and vector below, we use the ordered state space $S = (000, 001, 101, 110, 010, 011, 111, 101)$.

a. $P = \begin{pmatrix} 0 & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & \frac{2}{5} \\ \frac{1}{4} & 0 & 0 & 0 & 0 & \frac{3}{8} & 0 & \frac{3}{9} \\ 0 & \frac{1}{4} & 0 & 0 & \frac{3}{8} & 0 & \frac{3}{8} & 0 \\ 0 & 0 & \frac{1}{4} & 0 & 0 & \frac{3}{8} & 0 & \frac{3}{8} \\ 0 & 0 & 0 & \frac{1}{4} & \frac{3}{8} & 0 & \frac{3}{8} & 0 \end{pmatrix}$

b. $f_2 = \left(\frac{3}{32}, \frac{3}{32}, \frac{3}{32}, \frac{3}{32}, \frac{5}{32}, \frac{5}{32}, \frac{5}{32}, \frac{5}{32}\right)$

☑ 12.10. For the matrix and vectors below, we use the ordered state space is $S = (a, b, c, d)$.

a. The chain is aperiodic since the graph is not bipartite.

b. $f = \left(\frac{1}{7}, \frac{2}{7}, \frac{3}{14}, \frac{5}{14}\right)$

c. $\mu = \left(7, \frac{7}{2}, \frac{14}{3}, \frac{14}{5}\right)$

d. $P^n \to \begin{pmatrix} \frac{1}{7} & \frac{2}{7} & \frac{3}{14} & \frac{5}{14} \\ \frac{1}{7} & \frac{2}{7} & \frac{3}{14} & \frac{5}{14} \\ \frac{1}{7} & \frac{2}{7} & \frac{3}{14} & \frac{5}{14} \\ \frac{1}{7} & \frac{2}{7} & \frac{3}{14} & \frac{5}{14} \end{pmatrix}$ as $n \to \infty$

☑ 12.11. For the matrix and vector below, we use the ordered state space $S = (000, 001, 101, 110, 010, 011, 111, 101)$.

a. The chain has period 2 since the graph is bipartite. The cyclic classes are $\{000, 011, 110, 101\}$ (bit strings with an even number of 1's) and $\{010, 001, 100, 111\}$ (bit strings with an odd number of 1's).

b. $f = \left(\frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right)$

c. $\mu = (12, 12, 12, 12, 6, 6, 6, 6)$

d. $$P^{2n} \to \begin{pmatrix} \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} \end{pmatrix} \text{ as } n \to \infty$$

e. $$P^{2n+1} \to \begin{pmatrix} 0 & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{3} & 0 & \frac{1}{3} & 0 \end{pmatrix} \text{ as } n \to \infty$$

12.14. A conductance function c is $c(x, x-1) = \binom{m-1}{x-1}$, $c(x, x+1) = \binom{m-1}{x}$ for $x \in \{0, 1, ..., m\}$.
