4. Tests in the Two-Sample Normal Model

Preliminaries

In this section, we will study hypothesis tests in the two-sample normal model and in the bivariate normal model. This section parallels the section on Estimation in the Two Sample Normal Model in the chapter on Interval Estimation.

The Two-Sample Normal Model

Suppose that $\boldsymbol{X} = (X_1, X_2, ..., X_m)$ is a random sample of size m from the normal distribution with mean μ and standard deviation σ, and that $\boldsymbol{Y} = (Y_1, Y_2, ..., Y_n)$ is a random sample of size n from the normal distribution with mean ν and standard deviation τ. Moreover, suppose that the samples $\boldsymbol{X}$and $\boldsymbol{Y}$ are independent.

This type of situation arises frequently when the random variables represent a measurement of interest for the objects of the population, and the samples correspond to two different treatments. For example, we might be interested in the blood pressure of a certain population of patients. The $\boldsymbol{X}$ vector records the blood pressures of a control sample, while the $\boldsymbol{Y}$ vector records the blood pressures of the sample receiving a new drug. Similarly, we might be interested in the yield of an acre of corn. The $\boldsymbol{X}$ vector records the yields of a sample receiving one type of fertilizer, while the $\boldsymbol{Y}$ vector records the yields of a sample receiving a different type of fertilizer.

Usually our interest is in a comparison of the parameters (either the mean or variance) for the two sampling distributions. In this section we will construct tests for the ratio of the variances and for the difference of the means. As with previous estimation problems we have studied, the procedures vary depending on what parameters are known or unknown. Also as before, key elements in the construction of the tests are the sample means and sample variances and the special properties of these statistics when the sampling distribution is normal.

Notation

We will use the following notation for a generic sample $\boldsymbol{U} = (U_1, U_2, ..., U_k)$ and where a is a real number:

$$M(\boldsymbol{U}) = \frac{1}{k} \sum_{i=1}^{k} U_i, \quad W^2(\boldsymbol{U}, a) = \frac{1}{k} \sum_{i=1}^{k} (U_i - a)^2, \quad S^2(\boldsymbol{U}) = \frac{1}{k-1} \sum_{i=1}^{k} (U_i - M(\boldsymbol{U}))^2$$

Tests of the Ratio of the Variances

Tests When the Means are Known

We will first consider tests for the ratio of the variances $\frac{\tau^2}{\sigma^2}$ under the assumption that the means μ and ν are known. Usually, of course, this is an unrealistic assumption, but it is a good place to start because the analysis is fairly easy. Our basic test statistic is

$$F(\boldsymbol{X}, \boldsymbol{Y}, \rho) = \frac{W^2(\boldsymbol{X}, \mu)}{W^2(\boldsymbol{Y}, \nu)} \rho$$

where ρ is a conjectured value of the ratio of the variances.

1. Show that $F\left(\boldsymbol{X}, \boldsymbol{Y}, \frac{\tau^2}{\sigma^2}\right)$ has the F distribution with m degrees of freedom in the numerator and n degrees of freedom in the denominator.

Now for $p \in (0, 1)$ and for $m > 0$ and $n > 0$, let $f_{m,n}(p)$ denote the quantile of order p for the F distribution with m degrees of freedom in the numerator and n degrees of freedom in the denominator. For selected values of m, n, and p, $f_{m,n}(p)$ can be computed using the quantile applet or from most statistical software packages.

2. Show that the following tests have significance level α:

a. Reject H_0: $\frac{\tau^2}{\sigma^2} = \rho$ versus H_1: $\frac{\tau^2}{\sigma^2} \neq \rho$ if and only if $F(\boldsymbol{X}, \boldsymbol{Y}, \rho) \geq f_{m,n}\left(1 - \frac{\alpha}{2}\right)$ or $F(\boldsymbol{X}, \boldsymbol{Y}, \rho) \leq f_{m,n}\left(\frac{\alpha}{2}\right)$
b. Reject H_0: $\frac{\tau^2}{\sigma^2} \leq \rho$ versus H_1: $\frac{\tau^2}{\sigma^2} > \rho$ if and only if $F(\boldsymbol{X}, \boldsymbol{Y}, \rho) \leq f_{m,n}(\alpha)$
c. Reject H_0: $\frac{\tau^2}{\sigma^2} \geq \rho$ versus H_1: $\frac{\tau^2}{\sigma^2} < \rho$ if and only if $F(\boldsymbol{X}, \boldsymbol{Y}, \rho) \geq f_{m,n}(1 - \alpha)$

3. For each of the tests in Exercise 2, show that we fail to reject H_0 at significance level α if and only if ρ_0 is in the corresponding $1 - \alpha$ level confidence interval.

Tests When the Means are Unknown

Next we will consider tests for the ratio of the variances $\frac{\tau^2}{\sigma^2}$ under the more realistic assumption that the means μ and ν are unknown. In this case, our basic test statistic is

$$F(\boldsymbol{X}, \boldsymbol{Y}, \rho) = \frac{S^2(\boldsymbol{X})}{S^2(\boldsymbol{Y})} \rho$$

where once again, ρ is a conjectured value of the ratio of the variances.

4. Show that $F\left(\boldsymbol{X}, \boldsymbol{Y}, \frac{\tau^2}{\sigma^2}\right)$ has the F distribution with $m - 1$ degrees of freedom in the numerator and

$n - 1$ degrees of freedom in the denominator.

5. Show that the following tests have significance level α:

a. Reject $H_0: \frac{\tau^2}{\sigma^2} = \rho$ versus $H_1: \frac{\tau^2}{\sigma^2} \neq \rho$ if and only if $F(\boldsymbol{X}, \boldsymbol{Y}, \rho) \geq f_{m-1, n-1}\left(1 - \frac{\alpha}{2}\right)$ or $F(\boldsymbol{X}, \boldsymbol{Y}, \rho) \leq f_{m-1, n-1}\left(\frac{\alpha}{2}\right)$

b. Reject $H_0: \frac{\tau^2}{\sigma^2} \leq \rho$ versus $H_1: \frac{\tau^2}{\sigma^2} > \rho$ if and only if $F(\boldsymbol{X}, \boldsymbol{Y}, \rho) \leq f_{m-1, n-1}(\alpha)$

c. Reject $H_0: \frac{\tau^2}{\sigma^2} \geq \rho$ versus $H_1: \frac{\tau^2}{\sigma^2} < \rho$ if and only if $F(\boldsymbol{X}, \boldsymbol{Y}, \rho) \geq f_{m-1, n-1}(1 - \alpha)$

6. For each of the tests in Exercise 5, show that we fail to reject H_0 at significance level α if and only if ρ_0 is in the corresponding $1 - \alpha$ level confidence interval.

Tests of the Difference in the Means

Tests When the Variances are Known

Next we will consider tests for the difference in the means $\nu - \mu$ under the assumption that the standard deviations σ and τ are known. Once again, this is usually an unrealistic assumption, but it is a good place to start because the analysis is fairly easy. Our basic test statistic is

$$Z(\boldsymbol{X}, \boldsymbol{Y}, \delta) = \frac{(M(\boldsymbol{Y}) - M(\boldsymbol{X})) - \delta}{\sqrt{\frac{\sigma^2}{m} + \frac{\tau^2}{n}}}$$

where δ is a conjectured value of the difference in the means.

7. Show that $Z(\boldsymbol{X}, \boldsymbol{Y}, \delta)$ has the normal distribution with mean $(\nu - \mu) - \delta$ and variance 1.

As usual, for $p \in (0, 1)$, let $z(p)$ denote the quantile of order p for the standard normal distribution. For selected values of p, $z(p)$ can be obtained from the last row of the table of the t distribution, from the table of the standard normal distribution, from the quantile applet or from most statistical software packages. Recall also by symmetry that $z(1 - p) = -z(p)$.

8. Show that the following tests have significance level α:

a. Reject $H_0: \nu - \mu = \delta$ versus $H_1: \nu - \mu \neq \delta$ if and only if $Z(\boldsymbol{X}, \boldsymbol{Y}, \delta) \leq z\left(\frac{\alpha}{2}\right)$ or $Z(\boldsymbol{X}, \boldsymbol{Y}, \delta) \geq z\left(1 - \frac{\alpha}{2}\right)$

b. Reject $H_0: \nu - \mu \geq \delta$ versus $H_1: \nu - \mu < \delta$ if and only if $Z(\boldsymbol{X}, \boldsymbol{Y}, \delta) \leq z(\alpha)$

c. Reject $H_0: \nu - \mu \leq \delta$ versus $H_1: \nu - \mu > \delta$ if and only if $Z(\boldsymbol{X}, \boldsymbol{Y}, \delta) \geq z(1 - \alpha)$

9. For each of the tests in Exercise 8, show that we fail to reject H_0 at significance level α if and only if δ is

in the corresponding $1 - \alpha$ level confidence interval.

Tests When the Variances are Unknown

Finally we will consider tests for the difference in the means $\nu - \mu$ under the more realistic assumption that the standard deviations σ and τ are unknown. In this case, it is more difficult to find a suitable test statistic, but we can do the analysis in the special case that the standard deviations are the same. Thus, we will assume that $\sigma = \tau$, and the common value σ is unknown. This assumption is reasonable if there is an inherent variability in the measurement variables that does not change even when different treatments are applied to the objects in the population. Recall that the **pooled estimate** of the common variance σ^2 is

$$S^2(\boldsymbol{X}, \boldsymbol{Y}) = \frac{(m-1)\, S^2(\boldsymbol{X}) + (n-1)\, S^2(\boldsymbol{Y})}{m+n-2}$$

Our basic test statistic is

$$T(\boldsymbol{X}, \boldsymbol{Y}, \delta) = \frac{(M(\boldsymbol{Y}) - M(\boldsymbol{X})) - \delta}{S(\boldsymbol{X}, \boldsymbol{Y}) \sqrt{\frac{1}{m} + \frac{1}{n}}}$$

10. Show that if $T(\boldsymbol{X}, \boldsymbol{Y}, \nu - \mu)$ has the t distribution with $m + n - 2$ degrees of freedom,

As usual, for $k > 0$ and $p \in (0, 1)$, let $t_k(p)$ denote the quantile of order p for the t distribution with k degrees of freedom. For selected values of k and p, values of $t_k(p)$ are given in the table of the Student t distribution, from the quantile applet, or from most statistical software packages. Recall also that, by symmetry, $t_k(1 - p) = -t_k(p)$

11. Show that the following tests have significance level α:

a. Reject $H_0: \nu - \mu = \delta$ versus $H_1: \nu - \mu \neq \delta$ if and only if $T(\boldsymbol{X}, \boldsymbol{Y}, \delta) \le t_{m+n-2}\left(\frac{\alpha}{2}\right)$ or $T(\boldsymbol{X}, \boldsymbol{Y}, \delta) \ge t_{m+n-2}\left(1 - \frac{\alpha}{2}\right)$
b. Reject $H_0: \nu - \mu \ge \delta$ versus $H_1: \nu - \mu < \delta$ if and only if $T(\boldsymbol{X}, \boldsymbol{Y}, \delta) \le t_{m+n-2}(\alpha)$
c. Reject $H_0: \nu - \mu \le \delta$ versus $H_1: \nu - \mu > \delta$ if and only if $T(\boldsymbol{X}, \boldsymbol{Y}, \delta) \ge t_{m+n-2}(1 - \alpha)$

12. For each of the tests in Exercise 11, show that we fail to reject H_0 at significance level α if and only if δ_0 is in the corresponding $1 - \alpha$ level confidence interval.

Tests in the Bivariate Normal Model

In this subsection, we consider a model that is superficially similar to the two-sample normal model, but is actually much simpler. Suppose that

$$((X_1, Y_1), (X_2, Y_2), ..., (X_n, Y_n))$$

is a random sample of size n from the bivariate normal distribution of (X, Y) with $\mathbb{E}(X) = \mu$, $\mathbb{E}(Y) = \nu$, $\text{var}(X) = \sigma^2$, $\text{var}(Y) = \tau^2$, and $\text{cov}(X, Y) = \delta$.

Thus, instead of a *pair of samples*, we have a *sample of pairs*. This type of model frequently arises in **before and after experiments**, in which a measurement of interest is recorded for a sample of n objects from the population, both before and after a treatment. For example, we could record the blood pressure of a sample of n patients, before and after the administration of a certain drug.

We will use our usual notation for the sample means and variances of $\boldsymbol{X} = (X_1, X_2, ..., X_n)$ and $\boldsymbol{Y} = (Y_1, Y_2, ..., Y_n)$. Recall also that the sample covariance of $(\boldsymbol{X}, \boldsymbol{Y})$, is

$$S(\boldsymbol{X}, \boldsymbol{Y}) = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - M(\boldsymbol{X}))(Y_i - M(\boldsymbol{Y}))$$

13. Show that $\boldsymbol{Y} - \boldsymbol{X} = (Y_1 - X_1, Y_2 - X_2, ..., Y_n - X_n)$ is a random sample of size n from the distribution of $Y - X$, which is normal distribution with

a. $\mathbb{E}(Y - X) = \nu - \mu$
b. $\text{var}(Y - X) = \sigma^2 + \tau^2 - 2\,\delta$

14. Show that

a. $M(\boldsymbol{Y} - \boldsymbol{X}) = M(\boldsymbol{Y}) - M(\boldsymbol{X})$
b. $S^2(\boldsymbol{Y} - \boldsymbol{X}) = S^2(\boldsymbol{X}) + S^2(\boldsymbol{Y}) - 2\,S(\boldsymbol{X}, \boldsymbol{Y})$

The sample of differences $\boldsymbol{Y} - \boldsymbol{X}$ fits the normal model for a single variable. The section on Tests in the Normal Model could be used to perform tests for the parameters $(\nu - \mu, \sigma^2 + \tau^2 - 2\,\delta)$.

Computational Exercises

15. A new drug is being developed to reduce a certain blood chemical. A sample of 36 patients are given a placebo while a sample of 49 patients are given the drug. The statistics (in mg) are $m_1 = 87$, $s_1 = 4$, $m_2 = 63$, $s_2 = 6$. Test the following at the 10% significance level:

a. $H_0\colon\ \sigma_1 = \sigma_2$ versus $H_1\colon\ \sigma_1 \neq \sigma_2$.
b. $H_0\colon\ \mu_1 \le \mu_2$ versus $H_1\colon\ \mu_1 > \mu_2$. (assuming that $\sigma_1 = \sigma_2$).
c. Based on (b), is the drug effective?

16. A company claims that an herbal supplement improves intelligence. A sample of 25 persons are given a standard IQ test before and after taking the supplement. The before and after statistics are $m_1 = 105$,

$s_1 = 13$, $m_2 = 110$, $s_2 = 17$, $s_{1,2} = 190$. At the 10% significance level, do you believe the company's claim?

17. In Fisher's iris data, consider the petal length variable for the samples of Versicolor and Virginica irises. Test the following at the 10% significance level:

a. $H_0: \sigma_1 = \sigma_2$ versus $H_1: \sigma_1 \neq \sigma_2$.
b. $H_0: \mu_1 \leq \mu_2$ versus $H_1: \mu_1 > \mu_2$. (assuming that $\sigma_1 = \sigma_2$).

18. A plant has two machines that produce a circular rod whose diameter (in cm) is critical. A sample of 100 rods from the first machine as mean 10.3 and standard deviation 1.2. A sample of 100 rods from the second machine has mean 9.8 and standard deviation 1.6. Test the following hypotheses at the 10% level.

a. $H_0: \sigma_1 = \sigma_2$ versus $H_1: \sigma_1 \neq \sigma_2$.
b. $H_0: \mu_1 = \mu_2$ versus $H_1: \mu_1 \neq \mu_2$. (assuming that $\sigma_1 = \sigma_2$).