1. Introduction

Gambling and Probability

Games of chance are among the oldest of human inventions. The use of a certain type of animal heel bone (called the **astragalus**) as a crude die dates to about 3600 BCE. The modern six-sided die dates to 2000 BCE, and the term *bones* is used as a slang expression for dice to this day (as in *roll the bones*). It is because of these ancient origins, by the way, that we use the die as the fundamental symbol in this project.

Gambling is intimately interwoven with the development of probability as a mathematical theory. Most of the early development of probability, in particular, was stimulated by special gambling problems, such as

- DeMere's problem
- Pepy's problem
- the problem of points
- the Petersburg problem

Some of the very first books on probability theory were written to analyze games of chance, for example *Liber de Ludo Aleae* (The Book on Games of Chance), by Girolamo Cardano, and *Essay d' Analyse sur les Jeux de Hazard* (Analytical Essay on Games of Chance), by Pierre-Remond Montmort. Gambling problems continue to be a source of interesting and deep problems in probability to this day (see the discussion of Red and Black for an example).

Of course, it is important to keep in mind that breakthroughs in probability, even when they are originally motivated by gambling problems, are often profoundly important in many areas of the natural sciences, the social sciences, law, and medicine. Also, games of chance provide some of the conceptually clearest and cleanest examples of random experiments, and thus their analysis can be very helpful to students of probability.

However, nothing in this chapter should be construed as encouraging you, gentle reader, to gamble. On the contrary, our analysis will show that, in the long run, only the gambling houses prosper. The gambler, inevitably, is a sad victim of the law of large numbers.

In this chapter we will study some interesting games of chance. Poker, poker dice, craps, and roulette are popular parlor and casino games. The Monty Hall problem, on the other hand, is interesting because of the controversy that it generated. The lottery is a basic way that many states and nations use to raise money (a voluntary tax, of sorts)

Terminology

Let us discuss some of the basic terminology that will be used in several sections of this chapter. Suppose that A is an event in a random experiment. The **mathematical odds** concerning A refer to the probability of A

. Specifically, if a and b are positive numbers, then by definition, the following are equivalent:

- the **odds in favor** of A are $a : b$.
- $\mathbb{P}(A) = \frac{a}{a+b}$
- the **odds against** A are $b : a$.
- $\mathbb{P}(A^c) = \frac{b}{a+b}$

In many cases, a and b can be given as integers with no common factors.

1. Similarly, suppose that $p \in [0, 1]$. Show that the following are equivalent:

a. $\mathbb{P}(A) = p$
b. The odds in favor of A are $p : 1 - p$
c. $\mathbb{P}(A^c) = 1 - p$
d. The odds against A are $1 - p : p$

On the other hand, the **house odds** of an event refer to the **payout** when a bet is made on the event. To say that a bet on event A pays $n : m$ means that if a gambler bets m units on A and A occurs, the gambler receives the m units back and an additional n units (for a net profit of n); if A does not occur, the gambler loses the bet of m units (for a net profit of $-m$). Equivalently, the gambler puts up m units (betting on A), the house puts up n units, (betting on A^c) and the winner takes the pot. Of course, it is usually not necessary for the gambler to bet exactly m; a smaller or larger is bet is scaled appropriately. Thus, if the gambler bets k units and wins, his payout is $k \frac{n}{m}$.

Naturally, our main interest is in the net **winnings** if we make a bet on an event. The following exercise gives the probability density function, mean, and variance for a unit bet. The expected value is particularly interesting, because by the law of large numbers, it gives the long term gain or loss, per unit bet.

2. Suppose that the odds in favor of event A are $a : b$ and that a bet on event A pays $n : m$. Let W denote the winnings from a unit bet on A. Show that

a. $\mathbb{P}(W = -1) = \frac{b}{a+b}$, $\mathbb{P}\left(W = \frac{n}{m}\right) = \frac{a}{a+b}$
b. $\mathbb{E}(W) = \frac{a\,n - b\,m}{m\,(a+b)}$
c. $\text{var}(W) = \frac{a\,b\,(n+m)^2}{m^2\,(a+b)^2}$

In particular, the expected value of the bet is zero if and only if $a\,n = b\,m$, positive if and only if $a\,n > b\,m$, and negative if and only if $a\,n = b\,m$. The first case means that the bet is **fair**, and occurs when the payoff is the same as the odds *against* the event. The second means that the bet is **favorable** to the gambler, and occurs when the payoff is greater that the odds against the event. The third case means that the bet is **unfair** to the gambler, and occurs when the payoff is less than the odds against the event. Unfortunately, all casino games fall into the third category.

Simulation

It's very easy to simulate a fair die with a random number. Recall that the **ceiling function** $\lceil x \rceil$ gives the smallest integer that is at least as large as x.

3. Suppose that U is uniformly distributed on the interval $(0, 1]$ (a random number). Show that $X = 6\lceil U \rceil$ is uniformly distributed on the set $\{1, 2, 3, 4, 5, 6\}$.

To see how to simulate a card hand, see the Introduction to Finite Sampling Models. A general method of simulating random variables is based on the quantile function.
