

4. Partial Orders

A **relation** \preceq on S that is reflexive, anti-symmetric, and transitive is said to be a **partial order** on S . As the name and notation suggest, a partial order is a type of ordering of the elements of S . Partial orders occur naturally in many areas of mathematics, including probability.

A partial order \preceq on a set S naturally gives rise to several other relations on S : \succeq , $<$, $>$, \perp , and \parallel :

- $x \succeq y$ if and only if $y \preceq x$, so \succeq is the inverse of \preceq .
- $x < y$ if and only if $x \preceq y$ and $x \neq y$.
- $x > y$ if and only if $y < x$, so $>$ is the inverse of $<$.
- $x \perp y$ if and only if $x \preceq y$ or $y \preceq x$ so that x and y are related in the partial order.
- $x \parallel y$ if and only if $\neg(x \perp y)$ so x and y are unrelated in the partial order. Note that \parallel is the complement of \perp , as sets of ordered pairs.

▣ 1. Show that the inverse of a partial order is also a partial order.

▣ 2. Show that if \preceq is a partial order on S and A is a subset of S , then the restriction of \preceq to A is a partial order on A .

A partial order \preceq on S is a **total order** or **linear order** if it satisfies the additional property that for every $x \in S$ and $y \in S$, either $x \preceq y$ or $y \preceq x$. Of course, the ordinary order \leq is a total order on the set of real numbers \mathbb{R} .

▣ 3. Suppose that S is a set. Show that the subset relation \subseteq is a partial order on $\mathcal{P}(S)$, the power set of S .

The following exercise characterizes relations that correspond to strict order.

▣ 4. Let S be a set. Show that \preceq is a partial order on S if and only if $<$ is transitive and irreflexive.

Sub-orders

Suppose that \preceq_1 and \preceq_2 are partial orders on a nonempty set S . Then \preceq_1 is a **sub-order** of \preceq_2 , or equivalently \preceq_2 is an **extension** of \preceq_1 if and only if

$$x \preceq_1 y \Rightarrow x \preceq_2 y$$

Thus, as sets of ordered pairs, \preceq_1 is a subset of \preceq_2 .

▣ 5. Let $|$ denote the division relation on the set of positive integers \mathbb{N}_+ . That is, $m|n$ if and only if there exists

$k \in \mathbb{N}_+$ such that $n = km$. Show that $|$ is a partial order on \mathbb{N}_+ , and $|$ is a sub-order of the ordinary order \leq .

Hasse Diagrams

Suppose that \preceq is a partial order on a set S . For $x \in S$ and $y \in S$, y is said to **cover** x if $x < y$ but no element $z \in S$ satisfies $x < z < y$. If S is finite, the **covering relation** completely determines the partial order. Moreover, the **covering graph** or **Hasse graph** is the directed graph with vertex set S and directed edge set E , where $(x, y) \in E$ if and only if y covers x . Thus, $x < y$ if and only if there is a directed path in the graph from x to y . Hasse graphs are named for the German mathematician **Helmut Hasse**. The graphs are often drawn with the edges directed upward. In this way, the directions can be inferred without having to actually draw arrows.

6. Let $S = \{2, 3, 4, 6, 12\}$.

- Sketch the Hasse graph corresponding to the ordinary order \leq on S .
- Sketch the Hasse graph corresponding to the division partial order $|$ on S .

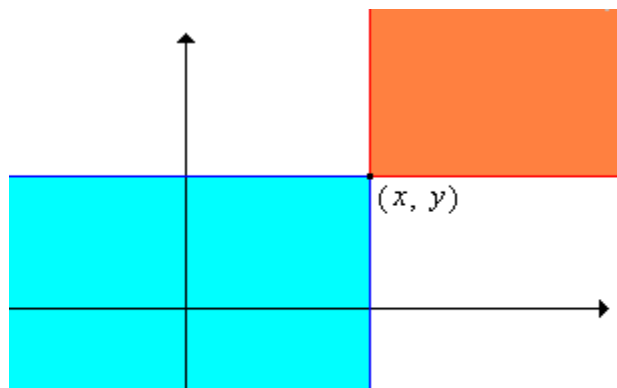


Orders on Product Spaces

7. Suppose that S and T are sets with partial orders \preceq_S and \preceq_T respectively. Define the relation \preceq on $S \times T$ by $(x, y) \preceq (z, w)$ if and only if $x \preceq_S z$ and $y \preceq_T w$.

- Show that \preceq is a partial order on $S \times T$, called, appropriately enough, the **product order**.
- Suppose that $(S, \preceq_S) = (T, \preceq_T)$. Show that if S has at least 2 elements, then \preceq is not a total order on S^2 .

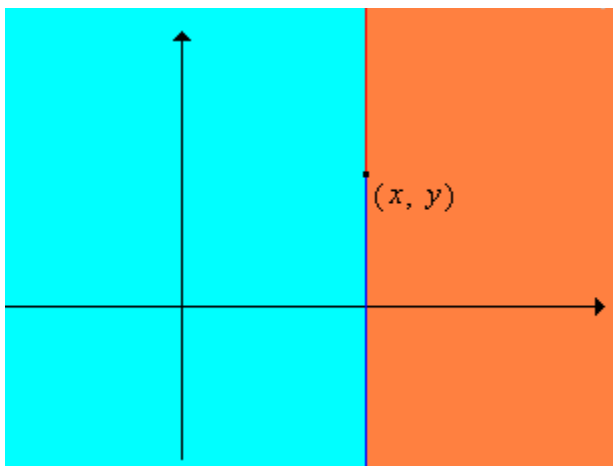
The picture below shows a point $(x, y) \in \mathbb{R}^2$. Suppose that \mathbb{R}^2 is given the product order associated with the ordinary order \leq on \mathbb{R} . Then the region shaded red is the set of points larger than (x, y) . The region shaded blue is the set of points smaller than (x, y) . The region shaded white is the set of points that are not comparable with (x, y) .





8. Suppose again that S and T are sets with partial orders \preceq_S and \preceq_T respectively. Define the relation \preceq on $S \times T$ by $(x, y) \preceq (z, w)$ if and only if either $x \preceq_S z$, or $x = z$ and $y \preceq_T w$
- Show that \preceq is a partial order on $S \times T$, called the **lexicographic order** or **dictionary order**.
 - Show that if \preceq_S and \preceq_T are total orders on S and T , respectively, then \preceq is a total order on $S \times T$.

The picture below shows a point $(x, y) \in \mathbb{R}^2$. Suppose that \mathbb{R}^2 is given the lexicographic order associated with the ordinary order \leq on \mathbb{R} . Then the region shaded red is the set of points larger than (x, y) . The region shaded blue is the set of points smaller than (x, y) .



Lexicographic orders are not as obscure as you might think. Many standard partially ordered sets can be expressed as lexicographic products.

9. A real number $x \in \mathbb{R}$ can be uniquely expressed in terms of its integer part and remainder:

$$x = n + t$$

where $n = \lfloor x \rfloor$ and $t = x - n = x - \lfloor x \rfloor$. In this way, \mathbb{R} can be associated with the product space $\mathbb{Z} \times [0, 1)$. Show that (\mathbb{R}, \leq) corresponds to the lexicographic product of (\mathbb{Z}, \leq) with $([0, 1), \leq)$, where of course \leq is the ordinary order for real numbers.

Monotone Sets and Functions

Partial orders form a natural setting for increasing and decreasing sets and functions. Suppose first that \preceq is a partial order on a set S and that $A \subseteq S$. Then

- A is **increasing** if $x \in A$ and $x \preceq y$ imply $y \in A$
- A is **decreasing** if $y \in A$ and $x \preceq y$ imply $x \in A$

Suppose now S is a set with partial order \preceq_S and that T is another set with partial order \preceq_T . Finally suppose that $f : S \rightarrow T$. Then

- f is **increasing** if and only if $x \preceq_S y \Rightarrow f(x) \preceq_T f(y)$
- f is **decreasing** if and only if $x \preceq_S y \Rightarrow f(x) \succeq_T f(y)$
- f is **strictly increasing** if and only if $x \prec_S y \Rightarrow f(x) \prec_T f(y)$
- f is **strictly decreasing** if and only if $x \prec_S y \Rightarrow f(x) \succ_T f(y)$

10. Suppose that \preceq is a partial order on a set S and that $A \subseteq S$. Recall the definition of the **indicator function** $\mathbf{1}(A)$ associated with A . Show that

- A is increasing if and only if $\mathbf{1}(A)$ is increasing.
- A is decreasing if and only if $\mathbf{1}(A)$ is decreasing.

In a sense, the subset partial order is universal--every partially ordered set is isomorphic to (\mathcal{S}, \subseteq) for some collection of sets \mathcal{S}

11. Suppose that \preceq is a partial order on a set S . For each $x \in S$, let $A(x) = \{u \in S : u \preceq x\}$, and then let $\mathcal{S} = \{A(x) : x \in S\}$. Show that the function A is one-to-one and satisfies

$$x \preceq y \text{ if and only if } A(x) \subseteq A(y)$$

Extremal Elements

Suppose again that \preceq is a partial order on a set S and that $A \subseteq S$. Various types of **extremal elements** of A play important roles.

- An element $a \in A$ is the **minimum** element of A if and only if $a \preceq x$ for every $x \in A$.
- An element $a \in A$ is a **minimal** element of A if and only if no $x \in A$ satisfies $x \prec a$.
- An element $b \in A$ is the **maximum** element of A if and only if $b \succeq x$ for every $x \in A$.
- An element $b \in A$ is a **maximal** element of A if and only if no $x \in A$ satisfies $x \succ b$.

In general, a set can have several maximal and minimal elements (or none).

12. Show that the minimum and maximum elements of A , if they exist, are unique. They are denoted $\min(A)$ and $\max(A)$, respectively.

Minimal, maximal, minimum, and maximum elements of a set must belong to that set. The following definitions relate to upper and lower bounds of a set, which do not have to belong to the set.

- An element $u \in S$ is a **lower bound** for A if and only if $u \preceq x$ for every $x \in A$.
- An element $v \in S$ is an **upper bound** for A if and only if $v \succeq x$ for every $x \in A$.
- The **greatest lower bound** or **infimum** of A , if it exists, is the maximum of the set of lower bounds of A .

- The **least upper bound** or **supremum** of A , if it exists, is the minimum of the set of upper bounds of A .

By the [Exercise 10](#), the greatest lower bound of A is unique, if it exists. It is denoted $\text{glb}(A)$ or $\text{inf}(A)$.

Similarly, the least upper bound of A is unique, if it exists, and is denoted $\text{lub}(A)$ or $\text{sup}(A)$.

13. Consider the ordinary order \leq on the set of real numbers \mathbb{R} , and let $A = [a, b)$ where $a < b$. Find each of the following that exist:

- The set of minimal elements of A
- The set of maximal elements of A
- $\text{min}(A)$
- $\text{max}(A)$
- The set of lower bounds of A
- The set of upper bounds of A
- $\text{inf}(A)$
- $\text{sup}(A)$



14. Consider the division partial order $|$ on the set of positive integers \mathbb{N}_+ and let A be a non-empty subset of \mathbb{N}_+ .

- Show that if A is infinite then $\text{sup}(A)$ does not exist. If A is finite then $\text{sup}(A)$ is the **least common multiple** of A , usually denoted $\text{lcm}(A)$ in this context.
- Show that $\text{inf}(A)$ is the **greatest common divisor** of A , usually denoted $\text{gcd}(A)$ in this context.

15. Again consider the division partial order $|$ on the set of positive integers \mathbb{N}_+ and let $A = \{2, 3, 4, 6, 12\}$. Find each of the following that exist:

- The set of minimal elements of A
- The set of maximal elements of A
- $\text{min}(A)$
- $\text{max}(A)$
- The set of lower bounds of A
- The set of upper bounds of A
- $\text{inf}(A)$
- $\text{sup}(A)$



16. Let S be a set and consider the subset partial order \subseteq on $\mathcal{P}(S)$, the power set of S . Let \mathcal{A} be a non-empty subset of $\mathcal{P}(S)$, that is, a non-empty collection of subsets of S . Show that

- a. $\inf(\mathcal{A}) = \bigcap \mathcal{A}$
 b. $\sup(\mathcal{A}) = \bigcup \mathcal{A}$

Suppose that S is a set and that $f : S \rightarrow S$. An element $z \in S$ is said to be a **fixed point** of f if $f(z) = z$. The following exercise explores a basic fixed point theorem for a partially ordered set. The theorem is important in the study of **cardinality**.

17. Suppose that \leq is a partial order on a set S with the property that $\sup(A)$ exists for every $A \subseteq S$. If f is an increasing function from S into S , then f has a fixed point. The following steps outline the proof:

- a. Let $A = \{x \in S : x \leq f(x)\}$ and let $z = \sup(A)$
 b. If $x \in A$ then $x \leq z$ so $x \leq f(x) \leq f(z)$,
 c. From (b), $f(z)$ is an upper bound of A so $z \leq f(z)$
 d. But then $f(z) \leq f(f(z))$ so $f(z) \in A$
 e. Hence $f(z) \leq z$
 f. Finally conclude that $f(z) = z$

Limits of Sequences of Real Numbers

Suppose that (a_1, a_2, \dots) is a sequence of real numbers.

18. Show that $\inf\{a_n, a_{n+1}, \dots\}$ is increasing in n

The limit of the sequence in the last exercise is the **limit inferior** of the original sequence:

$$\liminf_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \inf\{a_n, a_{n+1}, \dots\}$$

19. Show that $\sup\{a_n, a_{n+1}, \dots\}$ is decreasing in n

The limit of the sequence in the last exercise is the **limit superior** of the original sequence:

$$\limsup_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \sup\{a_n, a_{n+1}, \dots\}$$

Note that $\liminf_{n \rightarrow \infty} a_n \leq \limsup_{n \rightarrow \infty} a_n$ and equality holds if and only if $\lim_{n \rightarrow \infty} a_n$ exists (and is the common value).