Answers to Selected Exercises

4. Expected Value

---

1. Definitions and Properties

1.15.

b. $\mathbb{E}(X^n) = \frac{b^{n+1} - a^{n+1}}{(n+1)(b-a)}$

1.17. $\frac{2}{\pi}$

1.18. Let $Y = X^2$.

a. $g(y) = \begin{cases} \frac{1}{4} y^{-\frac{1}{2}}, & 0 < y < 1 \\ \frac{1}{8} y^{-\frac{1}{2}}, & 1 < y < 9 \end{cases}$

b. $\mathbb{E}(Y) = \frac{7}{3}$

c. $\mathbb{E}(Y) = \frac{7}{3}$

1.19.

a. $\mathbb{E}(Y) = 7$

b. $\mathbb{E}(M) = \frac{7}{2}$

c. $\mathbb{E}(Z) = \frac{49}{4}$

d. $\mathbb{E}(U) = \frac{101}{36}$

e. $\mathbb{E}(V) = \frac{19}{4}$

1.21.

a. $\mathbb{E}(Y) = 7$

b. $\mathbb{E}(M) = \frac{7}{2}$

c. $\mathbb{E}(Z) = \frac{49}{4}$
d. $\mathbb{E}(U) = \frac{77}{32}$
e. $\mathbb{E}(V) = \frac{147}{32}$

1.33. $\mathbb{E}(T | T > t) = t + \frac{1}{r}$

1.36.

a. Mean $\frac{3}{5}$
b. Mode $\frac{2}{3}$
c. Approximate median 0.614.

1.38.

a. $\mathbb{E}(V) = \frac{8}{21}\pi$
b. $\mathbb{E}(A) = \frac{8}{5}\pi$
c. $\mathbb{E}(C) = \frac{6}{5}\pi$

1.39.

a. Mean $\frac{1}{2}$
b. Median $\frac{1}{2}$

1.47.

a. $\frac{7}{12}$
b. $\frac{17}{72}$
c. $\frac{5}{6}$
d. $\frac{1}{3}$

1.48. $\mathbb{E}(3X + 4Y - 7) = 0$

1.49. $\mathbb{E}((3X - 4)(2Y + 7)) = 33$

1.50. Let N denote the number of ducks killed. $\mathbb{E}(N) = 10\left(1 - \left(\frac{9}{10}\right)^5\right) = 4.095$

1.59.

a. $\mathbb{E}(X) = \frac{1}{r}$
b. $e^{-r t} < \frac{1}{r t}, t > 0$

1.60.

a. $\mathbb{E}(W) = \frac{1}{p}$
b. $(1 - p)^{n-1} < \frac{1}{n\,p}, n \in \mathbb{N}_+$
c. $\mathbb{E}(W | W \text{ is even}) = \frac{2\,(1-p)^2}{p\,(2-p)^3}$

1.61.

a. $\mathbb{E}(X) = \frac{a}{a-1}$
b. $\mathbb{E}\left(\frac{1}{X}\right) = \frac{a}{a+1}$
d. $\frac{a}{a+1} > \frac{a-1}{a}$

1.62.

a. $\mathbb{E}\left(X^2 + Y^2\right) = \frac{5}{6}$
b. $\mathbb{E}(X)^2 + \mathbb{E}(Y)^2 = \frac{53}{72}$

2. Variance and Higher Moments

2.13. Let X denote the die score.

a. $\mathbb{E}(X) = \frac{7}{2}$
b. $\text{var}(X) = \frac{35}{12}$
c. $\text{sd}(X) \approx 1.708$

2.15. Let X denote the die score.

a. $\mathbb{E}(X) = \frac{7}{2}$
b. $\text{var}(X) = \frac{15}{4}$
c. $\text{sd}(X) \approx 1.936$

2.21. $\mathbb{E}(Y) = \frac{4}{3}$, $\text{sd}(Y) = \frac{2}{3}$, $k = 2$

a. $\mathbb{P}(|Y - \mathbb{E}(Y)| \geq k\,\text{sd}(Y)) = \frac{1}{16}$
b. $\frac{1}{k^2} = \frac{1}{4}$

2.24. $\mathbb{E}(X) = \frac{1}{r}$, $\text{sd}(Y) = \frac{1}{r}$, $k = 2$

a. $\mathbb{P}(|Y - \mathbb{E}(Y)| \geq k\,\text{sd}(Y)) = e^{-(k+1)}$
b. $\frac{1}{k^2}$

2.30.

a. $\mathbb{E}(X) = \frac{1}{2}$, $\text{var}(X) = \frac{1}{20}$,
b. $\mathbb{E}(X) = \frac{3}{5}$, $\text{var}(X) = \frac{1}{25}$,
c. $\mathbb{E}(X) = \frac{2}{5}$, $\text{var}(X) = \frac{1}{25}$,
d. $\mathbb{E}(X) = \frac{1}{2}$, $\text{var}(X) = \frac{1}{8}$,

2.31.

a. $\text{var}(3\,X - 2) = 36$
b. $\mathbb{E}\left(X^2\right) = 29$

2.33. Marilyn's standard score is $z = 8.53$

2.37.

a. $\text{skew}(X) = 0$
b. $\text{kurt}(X) = \frac{1296}{5}\,\frac{1}{(b-a)^4}$

2.38.

a. $\text{skew}(X) = 2\,r^3$
b. $\text{kurt}(X) = 9\,r^4$

2.39.

a. $\text{skew}(X) = \frac{2\,(a+1)\,(a-2)^2\,(a-1)^3}{a^2\,(a-3)}$
b. $\text{kurt}(X) = \frac{3\,\left(3\,a^2+a+2\right)\,(a-2)^3\,(a-1)^4}{a^3\,(a-3)\,(a-4)}$

2.40.

a. $\text{skew}(X) = 0$
b. $\text{kurt}(X) = 3$

2.41.

a. $\mathbb{E}(X) = \frac{1}{2}$, $\text{var}(X) = \frac{1}{20}$, $\text{skew}(X) = 0$, $\text{kurt}(X) = \frac{15}{7}$
b. $\mathbb{E}(X) = \frac{3}{5}$, $\text{var}(X) = \frac{1}{25}$, $\text{skew}(X) = -\frac{2}{7}$, $\text{kurt}(X) = \frac{33}{14}$
c. $\mathbb{E}(X) = \frac{2}{5}$, $\text{var}(X) = \frac{1}{25}$, $\text{skew}(X) = \frac{2}{7}$, $\text{kurt}(X) = \frac{33}{14}$
d. $\mathbb{E}(X) = \frac{1}{2}$, $\text{var}(X) = \frac{1}{8}$, $\text{skew}(X) = 0$, $\text{kurt}(X) = 96$

2.47.

a. $\|X\|_k = \frac{1}{(k+1)^{1/k}}$

c. 1

2.48.

a. $\|X\|_k = \begin{cases} \left(\frac{a}{a-k}\right)^{1/k}, & k < a \\ \infty, & k \ge a \end{cases}$

c. ∞

2.49.

a. $\|X + Y\|_k = \left(\frac{2^{k+2} - 2}{(k+2)\,(k+3)}\right)^{1/k}$

b. $\|X\|_k + \|Y\|_k = 2\left(\frac{1}{k+2} + \frac{1}{2\,(k+1)}\right)^{1/k}$

2.57.

a. When $p < \frac{1}{2}$, the minimum of $\mathbb{E}(|X - t|)$ is p and occurs at $t = 0$

b. When $p = \frac{1}{2}$, the minimum of $\mathbb{E}(|X - t|)$ is $\frac{1}{2}$ and occurs for $t \in [0,\ 1]$

c. When $p > \frac{1}{2}$, the minimum of $\mathbb{E}(|X - t|)$ is $1 - p$ and occurs at $t = 1$

3. Covariance and Correlation

3.27.

a. $\text{cov}(X,\ Y) = 0$, $\text{cor}(X,\ Y) = 0$.

b. $\text{cov}(X,\ Y) = \frac{a^2}{9}$, $\text{cor}(X,\ Y) = \frac{1}{2}$.

c. $\text{cov}(X,\ Y) = 0$, $\text{cor}(X,\ Y) = 0$.

3.29.

a. $\text{cov}(X,\ Y) = \frac{1}{24}$

b. $\text{cor}(X,\ Y) = \frac{\sqrt{12}}{14}$.

c. $L(Y|X) = \frac{1}{2}\,X$

d. $L(X|Y) = \frac{2}{7} + \frac{6}{7}\,Y$

3.30.

a. $\text{cov}(X_1,\ X_2) = 0$, $\text{cor}(X_1,\ X_2) = 0$

b. $\text{cov}(X_1, Y) = \frac{35}{12}$, $\text{cor}(X_1, Y) = \frac{1}{\sqrt{2}} = 0.7071$
c. $\text{cov}(X_1, U) = \frac{35}{24}$, $\text{cor}(X_1, U) = 0.6082$
d. $\text{cov}(U, V) = \frac{1369}{1296}$, $\text{cor}(U, V) = \frac{1369}{2555} = 0.5358$
e. $\text{cov}(U, Y) = \frac{35}{12}$, $\text{cor}(U, Y) = 0.8601$

3.31. Let Y denote the sum of the dice scores and M the average of the dice scores.

a. $\mathbb{E}(Y) = n\frac{7}{2}$, $\text{var}(Y) = n\frac{35}{12}$
b. $\mathbb{E}(M) = \frac{7}{2}$, $\text{var}(M) = \frac{35}{12}\frac{1}{n}$

3.33. Let Y denote the sum of the dice scores and M the average of the dice scores.

a. $\mathbb{E}(Y) = n\frac{7}{2}$, $\text{var}(Y) = n\frac{15}{4}$
b. $\mathbb{E}(M) = \frac{7}{2}$, $\text{var}(M) = \frac{15}{4}\frac{1}{n}$

3.35.

a. $L(Y|X_1) = \frac{7}{2} + X_1$
b. $L(U|X_1) = \frac{7}{9} + \frac{1}{2}X_1$
c. $L(V|X_1) = \frac{49}{19} + \frac{1}{2}X_1$

3.43. $\text{cov}(2X - 5, 4Y + 2) = 24$
3.44. $\text{var}(2X + 3Y - 7) = 65$
3.45. $\text{var}(3X - 4Y + 5) = 182$
3.46.

a. $\text{cov}(A, B) = \frac{1}{24}$
b. $\text{cor}(A, B) \approx 0.1768$

3.47.

a. $\text{cov}(X, Y) = -\frac{1}{144}$
b. $\text{cor}(X, Y) = -\frac{1}{11} = -0.0909$
c. $L(Y|X) = \frac{7}{11} - \frac{1}{11}X$
d. $L(X|Y) = \frac{7}{11} - \frac{1}{11}Y$

3.48.

a. $\text{cov}(X, Y) = \frac{1}{48}$

b. $\text{cor}(X, Y) = \frac{5}{\sqrt{129}} = 0.4402$
c. $L(Y|X) = \frac{26}{43} + \frac{15}{43} X$
d. $L(X|Y) = \frac{5}{9} Y$

3.49.

c. $\text{cov}\left(X^2, Y\right) = \frac{7}{360}$
d. $\text{cor}\left(X^2, Y\right) = 0.448$
e. $L\left(Y \middle| X^2\right) = \frac{1255}{1902} + \frac{245}{634} X^2$
f. The predictor based on X^2 is slightly better than the predictor based on X.

3.50. Note that X and Y are independent.

a. $\text{cov}(X, Y) = 0$
b. $\text{cor}(X, Y) = 0$
c. $L(Y|X) = \frac{2}{3}$
d. $L(X|Y) = \frac{3}{4}$

3.51.

a. $\text{cov}(X, Y) = \frac{5}{336}$
b. $\text{cor}(X, Y) = 0.05423$
c. $L(Y|X) = \frac{30}{51} + \frac{20}{51} X$
d. $L(X|Y) = \frac{3}{4} Y$

3.52.

c. $\text{cov}\left(\sqrt{X}, Y\right) = \frac{10}{1001}$
d. $\text{cor}\left(\sqrt{X}, Y\right) = \frac{24}{169} \sqrt{14}$
e. $L\left(Y \middle| \sqrt{X}\right) = \frac{5225}{13182} + \frac{1232}{2197} \sqrt{X}$
f. The predictor based on X is slightly better than the predictor based on $\sqrt{X}$

3.59. $\langle X, Y \rangle = \frac{1}{3}$

a. $\|X\|_2 \, \|Y\|_2 = \frac{5}{12}$
b. $\|X\|_3 \, \|Y\|_{\frac{3}{2}} \approx 0.4248$

4. Generating Functions

4.32.

a. $M(s, t) = 2\frac{e^{s+t}-1}{s(s+t)} - 2\frac{e^t-1}{s\,t}$, $s \ne 0$, $t \ne 0$

b. $M_X(s) = 2\left(\frac{e^s}{s^2} - \frac{1}{s^2} - \frac{1}{s}\right)$, $s \ne 0$

c. $M_Y(t) = 2\frac{t\,e^t - e^t + 1}{t^2}$, $t \ne 0$

d. $M_{X+Y}(t) = \frac{e^{2t}-1}{t^2} - 2\frac{e^t-1}{t^2}$, $t \ne 0$

4.33.

a. $M(s, t) = \frac{e^{s+t}(-2\,s\,t+s+t)+e^s(s\,t-s-t)+s+t}{s^2\,t^2}$, $s \ne 0$, $t \ne 0$

b. $M_X(s) = \frac{3\,s\,e^s - 2\,e^s - s + 2}{2\,s^2}$, $s \ne 0$

c. $M_Y(t) = \frac{3\,t\,e^t - 2\,e^t - t + 2}{2\,t^2}$, $t \ne 0$

d. $M_{X+Y}(t) = \frac{2\left(e^{2t}(1-t)+e^t(t-2)+1\right)}{t^3}$, $t \ne 0$

5. Conditional Expected Value

5.21.

a. $L(Y|X) = \frac{7}{11} - \frac{1}{11}X$

b. $\mathbb{E}(Y|X) = \frac{3\,X+2}{6\,X+3}$

d. $\text{var}(Y) = \frac{11}{144} = 0.0764$

e. $\text{var}(Y)\left(1 - \text{cor}(X, Y)^2\right) = \frac{5}{66} = 0.0758$

f. $\text{var}(Y) - \text{var}(\mathbb{E}(Y|X)) = \frac{1}{12} - \frac{1}{144}\ln(3) = 0.0757$

5.22.

a. $L(Y|X) = \frac{26}{43} + \frac{15}{43}X$

b. $\mathbb{E}(Y|X) = \frac{5\,X^2+5\,X+2}{9\,X+3}$

d. $\text{var}(Y) = \frac{3}{80} = 0.0375$

e. $\text{var}(Y)\left(1 - \text{cor}(X, Y)^2\right) = \frac{13}{430} = 0.0302$

f. $\text{var}(Y) - \text{var}(\mathbb{E}(Y|X)) = \frac{1837}{21870} - \frac{512}{6561}\ln(2) = 0.0299$

5.23.

a. $L(Y|X) = \frac{2}{3}$

b. $\mathbb{E}(Y|X) = \frac{2}{3}$

d. $\text{var}(Y) = \frac{1}{18}$

e. $\text{var}(Y)\left(1 - \text{cor}(X, Y)^2\right) = \frac{1}{18}$

f. $\text{var}(Y) - \text{var}(\mathbb{E}(Y|X)) = \frac{1}{18}$

5.24.

a. $L(Y|X) = \frac{30}{51} + \frac{20}{51} X$

b. $\mathbb{E}(Y|X) = \frac{2\left(X^2+X+1\right)}{3\,(X+1)}$

d. $\text{var}(Y) = \frac{5}{252} = 0.0198$

e. $\text{var}(Y)\left(1 - \text{cor}(X, Y)^2\right) = \frac{5}{357} = 0.0140$

f. $\text{var}(Y) - \text{var}(\mathbb{E}(Y|X)) = \frac{292}{63} - \frac{20}{3}\ln(2) = 0.0139$

5.25. $\mathbb{E}\left(Y e^X - Z\sin(X)\middle|X\right) = X^3 e^X - \frac{\sin(X)}{1+X^2}$

5.26. $\mathbb{E}(Y|X) = \mathbb{E}(Y) = \frac{1}{2}(c+d)$

5.28. $\mathbb{E}(Y|X) = \frac{a+X}{2}$

5.30.

a. $\mathbb{E}(Y|X) = \frac{1}{2} X$

b. $\mathbb{E}(Y) = \frac{1}{4}$

c. $\text{var}(Y|X) = \frac{1}{12} X^2$

d. $\text{var}(Y) = \frac{7}{144}$

5.31.

a. $\mathbb{E}(Y|Y_1) = \frac{7}{2} + X_1$

b.

x	1	2	3	4	5	6
$\mathbb{E}(U|X_1 = x)$	1	$\frac{11}{6}$	$\frac{5}{2}$	3	$\frac{10}{3}$	$\frac{7}{2}$

c.

u	1	2	3	4	5	6
$\mathbb{E}(Y|U = u)$	$\frac{52}{11}$	$\frac{56}{9}$	$\frac{54}{7}$	$\frac{46}{5}$	$\frac{32}{3}$	12

d. $\mathbb{E}(X_2|X_1) = \frac{7}{2}$

5.32. $\mathbb{P}(H) = \frac{1}{2}$

5.36.

a. Given N, X has the binomial distribution with parameters N and $p = \frac{1}{2}$
b. $\mathbb{E}(X|N) = \frac{1}{2} N$
c. $\text{var}(X|N) = \frac{1}{4} N$
d. $\mathbb{E}(X) = \frac{7}{4}$
e. $\text{var}(X) = \frac{7}{3}$

5.38. Let Y denote the amount of money spent during the hour.

a. $\mathbb{E}(Y) = \$1000$
b. $\text{sd}(Y) \approx \$30.82$

5.39.

a. $\mathbb{E}(Y|N, V) = N\, V$
b. $\mathbb{E}(Y|N) = \frac{1}{2} N$
c. $\mathbb{E}(Y|V) = a\, V$
d. $\mathbb{E}(Y) = \frac{1}{2} a$
e. $\text{var}(Y|N, V) = N\, V\, (1 - V)$
f. $\text{var}(Y) = \frac{1}{12} a^2 + \frac{1}{2} a$

5.44. Let X denote the die score

a. $\mathbb{E}(X) = \frac{7}{2}$
b. $\text{var}(X) \approx 1.8634$

6. Expected Value and Covariance Matrices

6.18.

a. $\mathbb{E}(X, Y) = \left(\frac{7}{12}, \frac{7}{12}\right)$
b. $\text{VC}(X, Y) = \begin{pmatrix} \frac{11}{144} & \frac{-1}{144} \\ \frac{-1}{144} & \frac{11}{144} \end{pmatrix}$

6.19.

a. $\mathbb{E}(X, Y) = \left(\frac{5}{12}, \frac{3}{4}\right)$

b. $\mathrm{VC}(X, Y) = \begin{pmatrix} \frac{43}{720} & \frac{1}{48} \\ \frac{1}{48} & \frac{3}{80} \end{pmatrix}$

6.20.

a. $\mathbb{E}(X, Y) = \left(\frac{3}{4}, \frac{2}{3}\right)$

b. $\mathrm{VC}(X, Y) = \begin{pmatrix} \frac{3}{80} & 0 \\ 0 & \frac{1}{18} \end{pmatrix}$

6.21.

a. $\mathbb{E}(X, Y) = \left(\frac{5}{8}, \frac{5}{6}\right)$

b. $\mathrm{VC}(X, Y) = \begin{pmatrix} \frac{17}{448} & \frac{5}{336} \\ \frac{5}{336} & \frac{5}{252} \end{pmatrix}$

c. $L(Y|X) = \frac{10}{17} + \frac{20}{51} X$

d. $L\left(Y \middle| X, X^2\right) = \frac{49}{76} + \frac{10}{57} X + \frac{7}{38} X^2$

6.22.

a. $\mathbb{E}(X, Y, Z) = \left(\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\right)$

b. $\mathrm{VC}(X, Y, Z) = \begin{pmatrix} \frac{3}{80} & \frac{1}{40} & \frac{1}{80} \\ \frac{1}{40} & \frac{1}{20} & \frac{1}{40} \\ \frac{1}{80} & \frac{1}{40} & \frac{3}{80} \end{pmatrix}$

c. $L(Z|X, Y) = \frac{1}{2} + \frac{1}{2} Y$. Note that there is no X term.

d. $L(Y|X, Z) = \frac{1}{2} X + \frac{1}{2} Z$. Note that this is the midpoint of the interval $[X, Z]$

e. $L(X|Y, Z) = \frac{1}{2} Y$. Note that there is no Z term.

6.23.

a. $\mathbb{E}(X, Y) = \left(\frac{1}{2}, \frac{1}{4}\right)$

b. $\mathrm{VC}(X, Y) = \begin{pmatrix} \frac{1}{12} & \frac{1}{24} \\ \frac{1}{24} & \frac{7}{144} \end{pmatrix}$