Answers to Selected Exercises

3. Distributions

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1. Discrete Distributions

1.18. $f(y) = \frac{\binom{30}{y}\binom{20}{5-y}}{\binom{50}{5}}$, $y \in \{0, 1, 2, 3, 4, 5\}$.

a. $f(0) = 0.0073$, $f(1) = 0.0686$, $f(2) = 0.2341$, $f(3) = 0.3641$, $f(4) = 0.2587$, $f(5) = 0.0673$
b. mode: $y = 3$
c. $\mathbb{P}(Y > 3) = 0.3260$

1.20. $f(y) = \binom{5}{y}\left(\frac{3}{5}\right)^y\left(\frac{2}{5}\right)^{5-y}$, $y \in \{0, 1, 2, 3, 4, 5\}$

a. $f(0) = 0.0102$, $f(1) = 0.0768$, $f(2) = 0.2304$, $f(3) = 0.3456$, $f(4) = 0.2592$, $f(5) = 0.0778$
b. mode: $y = 3$
c. $\mathbb{P}(Y > 3) = 0.3370$

1.22. $f(y) = \binom{5}{y} 0.4^y\, 0.6^{5-y}$, $y \in \{0, 1, 2, 3, 4, 5\}$.

a. $f(0) = 0.0778$, $f(1) = 0.2592$, $f(2) = 0.3456$, $f(3) = 0.2304$, $f(4) = 0.0768$, $f(5) = 0.0102$
b. mode: $y = 2$
c. $\mathbb{P}(Y > 3) = 0.0870$

1.24.

a. $\mathbb{P}((X_1, X_2) = (x_1, x_2)) = \frac{1}{36}$, $(x_1, x_2) \in \{1, 2, 3, 4, 5, 6\}^2$
b. $\mathbb{P}(Y = y) = \frac{6-|7-y|}{36}$, $y \in \{2, 3, ..., 12\}$
c. $\mathbb{P}(U = u) = \frac{13-2u}{36}$, $u \in \{1, 2, 3, 4, 5, 6\}$
d. $\mathbb{P}(V = v) = \frac{2v-1}{36}$, $v \in \{1, 2, 3, 4, 5, 6\}$
e. $\mathbb{P}((U, V) = (u, v)) = \begin{cases} \frac{2}{36}, & u < v \\ \frac{1}{36}, & u = v \end{cases}$, $(u, v) \in \{1, 2, 3, 4, 5, 6\}^2$
f. $\mathbb{P}(U = u | Y = 8) = \begin{cases} \frac{2}{5}, & u = 2 \\ \frac{2}{5}, & u = 3 \\ \frac{1}{5}, & u = 4 \end{cases}$

1.26.

a. $\mathbb{P}(N = n) = \frac{1}{6}$ for $n \in \{1, 2, ..., 6\}$

b. $\mathbb{P}(\boldsymbol{X} = \boldsymbol{x}) = \frac{1}{6}\left(\frac{1}{2}\right)^n$ for $\boldsymbol{x} \in \{0, 1\}^n$ and $n \in \{1, 2, 3, 4, 5, 6\}$

c. $$\mathbb{P}(Y = y) = \begin{cases} \frac{63}{384}, & y = 0 \\ \frac{120}{384}, & y = 1 \\ \frac{99}{384}, & y = 2 \\ \frac{64}{384}, & y = 3 \\ \frac{29}{384}, & y = 4 \\ \frac{8}{384}, & y = 5 \\ \frac{1}{384}, & y = 6 \end{cases}$$

d. $\mathbb{P}(N = n | Y = 2) = \frac{64}{99}\binom{n}{2}\left(\frac{1}{2}\right)^n$, $n \in \{2, 3, 4, 5, 6\}$

1.28. Let V denote the probability of heads for the selected coin and Y the number of heads.

a. $$\mathbb{P}(V = v) = \begin{cases} \frac{5}{12}, & v = \frac{1}{2} \\ \frac{4}{12}, & v = \frac{1}{3} \\ \frac{3}{12}, & v = 1 \end{cases}$$

b. $$\mathbb{P}(Y = y) = \begin{cases} \frac{5311}{93312}, & y = 0 \\ \frac{16315}{93312}, & y = 1 \\ \frac{22390}{93312}, & y = 2 \\ \frac{17270}{93312}, & y = 3 \\ \frac{7355}{93312}, & y = 4 \\ \frac{24671}{93312}, & y = 5 \end{cases}$$

c. $$\mathbb{P}(V = v | X = 2) = \begin{cases} \frac{45}{169}, & v = \frac{1}{2} \\ \frac{16}{169}, & v = \frac{1}{3} \\ \frac{108}{169}, & v = 1 \end{cases}$$

1.29. Let X denote the die score.

$$\mathbb{P}(X = x) = \begin{cases} \frac{5}{24}, & x \in \{1, 6\} \\ \frac{7}{48}, & x \in \{2, 3, 4, 5\} \end{cases}$$

1.31.

a. $f(n) = p(1 - p)^{n-1}$, $n \in \mathbb{N}_+$

b. $\mathbb{P}(N > n) = (1 - p)^n$, $n \in \mathbb{N}_+$

c. $\mathbb{P}(N \text{ is even}) = \frac{1-p}{2-p}$

d. $f(n | N \text{ is even}) = p(2 - p)(1 - p)^{n-2}$, $n \in \{2, 4, ...\}$

1.33.

a. $\mathbb{P}(X = x) = \frac{\binom{13}{x}\binom{39}{5-x}}{\binom{52}{5}}, x \in \{0, 1, 2, 3, 4, 5\}$
b. $\mathbb{P}(Y = y) = \frac{\binom{13}{y}\binom{39}{5-y}}{\binom{52}{5}}, y \in \{0, 1, 2, 3, 4, 5\}$
c. $\mathbb{P}(X = x, Y = y) = \frac{\binom{13}{x}\binom{13}{y}\binom{26}{5-x-y}}{\binom{52}{5}}, (x, y) \in \{0, 1, 2, 3, 4, 5\}^2$

1.34.

a. $\mathbb{P}(N = n) = \frac{\binom{16}{n}\binom{36}{13-n}}{\binom{52}{13}}, n \in \{0, 1, ..., 13\}$

1.35.

a. $\mathbb{P}(X = x) = \frac{\binom{20}{x}\binom{480}{10-x}}{\binom{500}{10}}, y \in \{0, 1, ..., 10\}$
b. $\mathbb{P}(X \geq 1) = 1 - \frac{\binom{480}{10}}{\binom{500}{10}} \approx 0.3377$

1.36. Let X denote the line number and D the event that the item is defective.

a. $\mathbb{P}(D) = 0.037$
b. $\mathbb{P}(X = x | D) = \begin{cases} 0.541, & x = 1 \\ 0.405, & x = 2 \\ 0.054, & x = 3 \end{cases}$

1.37. $f(x_1, x_2, ..., x_n) = p^k (1 - p)^{n-k}$ for $(x_1, x_2, ..., x_n) \in \{0, 1\}^n$ where $k = x_1 + x_2 + \cdots + x_n$

1.38.

a. $g(k) = \binom{n}{k} p^k (1 - p)^{n-k}, k \in \{0, 1, ..., n\}$
b. $r(p) = \sum_{i=k}^{n} \binom{n}{i} p^i (1 - p)^{n-i}$

1.40.

a. mode: 2
b. $\mathbb{P}(N > 4) = 0.10882$

1.42.

b. Mode $n = 1$
c. $\mathbb{P}(N \leq 5) = \frac{5269}{600\,\pi^2}$

1.43.

b.

d	1	2	3	4	5	6	7	8	9
$f(d)$	0.3010	0.1761	0.1249	0.0969	0.0792	0.0669	0.0580	0.0512	0.0458

c. $\mathbb{P}(X \leq 3) = \log(4) \approx 0.6020$

1.44.

b. $\mathbb{P}(3 \le N \le 7) = \frac{5}{24}$

1.45.

a. $f(n) = \frac{1}{165} n (10 - n)$, $n \in \{1, 2, ..., 9\}$

b. mode $n = 5$

c. $\mathbb{P}(3 \le N \le 6) = \frac{94}{165}$

1.46.

a. $f(n) = \frac{1}{825} n^2 (10 - n)$, $n \in \{1, 2, ..., 9\}$

b. Mode $n = 7$

c. $\mathbb{P}(3 \le N \le 6) = \frac{428}{825}$

1.47.

a. $f(x, y) = \frac{1}{36}(x + y)$, $(x, y) \in \{1, 2, 3\}^2$

b. mode $(3, 3)$

c. $\mathbb{P}(X > Y) = \frac{2}{9}$

1.48.

a. $f(x, y) = \frac{1}{25} x y$, $(x, y) \in \{(1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)\}$

b. mode $(3, 3)$

c. $\mathbb{P}((X, Y) \in \{(1, 2), (1, 3), (2, 2), (2, 3)\}) = \frac{3}{5}$

1.49.

a.

r	3	4	5	6	8	9	10	11	12	14	15	20
$\mathbb{P}(R = r)$	$\frac{1}{30}$	$\frac{3}{30}$	$\frac{3}{30}$	$\frac{2}{30}$	$\frac{4}{30}$	$\frac{5}{30}$	$\frac{2}{30}$	$\frac{1}{30}$	$\frac{3}{30}$	$\frac{3}{30}$	$\frac{3}{30}$	$\frac{1}{30}$

b.

n	50	53	54	55	56	57	58	59	60	61
$\mathbb{P}(N = n)$	$\frac{1}{30}$	$\frac{1}{30}$	$\frac{1}{30}$	$\frac{4}{30}$	$\frac{4}{30}$	$\frac{3}{30}$	$\frac{9}{30}$	$\frac{3}{30}$	$\frac{2}{30}$	$\frac{2}{30}$

c.

r	3	4	6	8	9	11	12	14	15
$\mathbb{P}(R = r \mid N > 57)$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{3}{16}$	$\frac{3}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{3}{16}$	$\frac{2}{16}$

1.50. Gender G: 0 (female), 1 (male). Species S: 0 (tredecula), 1 (tredecim), 2 (tredecassini).

a.

i	0	1
$\mathbb{P}(G = i)$	$\frac{59}{104}$	$\frac{45}{104}$

b.

j	0	1	2
$\mathbb{P}(S = j)$	$\frac{44}{104}$	$\frac{6}{104}$	$\frac{54}{104}$

c.

$\mathbb{P}(G = i, S = j)$	i

		0	1
j	0	$\frac{16}{104}$	28 / 104
	1	$\frac{3}{104}$	$\frac{3}{104}$
	2	$\frac{40}{104}$	$\frac{14}{104}$

d.

i	0	1
$\mathbb{P}(G = i \mid W > 0.2)$	$\frac{31}{73}$	$\frac{42}{73}$

2. Continuous Distributions

2.5. $\mathbb{P}(T > 2) = e^{-1} \approx 0.3679$

2.7.

b. mode $\theta = \frac{\pi}{2}$

c. $\mathbb{P}\left(\Theta < \frac{\pi}{4}\right) = 1 - \frac{1}{\sqrt{2}} \approx 0.2929$

2.10. $\mathbb{P}(T > 3) = \frac{17}{2} e^{-3} \approx 0.4232$

2.12.

b. $f(x) = 12 x^2 (1 - x), 0 \le x \le 1$

c. mode $x = \frac{2}{3}$

d. $\mathbb{P}\left(\frac{1}{2} < X < 1\right) = \frac{11}{16}$

2.13.

b. $f(x) = \frac{1}{\pi \sqrt{x (1 - x)}}, \ 0 < x < 1$

c. $\mathbb{P}\left(0 < X < \frac{1}{4}\right) = \frac{1}{3}$

2.16. $\mathbb{P}(X > 2) = \frac{1}{4}$

2.18.

c. $\mathbb{P}(-1 < X < 1) = \frac{1}{2}$

2.22.

b. Mode $x = 0$

c. $\mathbb{P}(X > 0) = 1 - e^{-1} \approx 0.6321$

2.24.

c. $\mathbb{P}\left(\frac{1}{3} \le X \le \frac{1}{2}\right) = \frac{1}{2}\ln(2) - \frac{1}{3}\ln(3) + \frac{1}{6} \approx 0.147$

2.25.

a. $f(x) = 2 e^{-x} (1 - e^{-x})$ for $x \ge 0$

b. Mode $x = \ln(2)$

c. $\mathbb{P}(X > 1) = 2 e^{-1} - e^{-2} \approx 0.6004$

2.26.

b. $\mathbb{P}(Y \geq X) = \frac{1}{2}$
c. $f\left(x, y \middle| X < \frac{1}{2}, Y < \frac{1}{2}\right) = 8\,(x + y), \quad 0 < x < \frac{1}{2}, 0 < y < \frac{1}{2}$

2.27.

a. $f(x, y) = 2\,(x + y), 0 \leq x \leq y \leq 1$
b. $\mathbb{P}(Y \geq 2\,X) = \frac{5}{12}$

2.28.

a. $f(x, y) = 6\,x^2\,y, 0 \leq x \leq 1, 0 \leq y \leq 1$
b. $\mathbb{P}(Y \geq X) = \frac{2}{5}$

2.29.

a. $f(x, y) = 15\,x^2\,y, 0 \leq x \leq y \leq 1$
b. $\mathbb{P}(Y > 2\,X) = \frac{1}{8}$

2.30.

a. $f(x, y, z) = \frac{1}{3}\,(x + 2\,y + 3\,z), 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1$
b. $\mathbb{P}(X \leq Y \leq Z) = \frac{7}{36}$

2.31.

a. $f(x, y) = 2\,e^{-x}\,e^{-y}, 0 < x < y < \infty$
b. $\mathbb{P}(X + Y < 1) = 1 - 2\,e^{-1} \approx 0.2642$

2.33.

a. $\mathbb{P}(X > 0, Y > 0) = \frac{1}{4}$
b. $\mathbb{P}(X > 0, Y > 0) = \frac{1}{4}$
c. $\mathbb{P}(X > 0, Y > 0) = \frac{1}{4}$

2.35. $\mathbb{P}(X < Y < Z) = \frac{1}{6}$

2.36.

a. $\mathbb{P}(T > 30) = \frac{2}{3}$
b. $\mathbb{P}(T > 45 | T > 30) = \frac{1}{2}$

2.38. Empirical densities, based on simple partitions of the body weight range and body length range

a.

BW	(0, 0.1]	(0.1, 0.2]	(0.2, 0.3]	(0.3, 0.4]
Density	0.8654	5.8654	3.0769	0.1923

b.

BL	(15, 29]	(20, 25]	(25, 30]	(30, 35]
Density	0.0058	0.1577	0.0346	0.0019

c.

BW	(0, 0.1]	(0.1, 0.2]	(0.2, 0.3]	(0.3, 0.4]
Density $G = 0$	0.3390	4.4068	5.0847	0.1695

2.40.

c. $\mathbb{P}(Y > X) = \frac{1}{2}$

2.41.

c. $\mathbb{P}(Y > X) = \frac{1}{2}$

3. Mixed Distributions

3.10. $\mathbb{P}(X > 6) = \frac{13}{40}$

3.11. $\mathbb{P}(Y > X) = \frac{4}{9}$

3.12.

a. $\mathbb{P}(U < 1) = 1 - e^{-1} \approx 0.6321$
b. $\mathbb{P}(U = 2) = e^{-2} \approx 0.1353$

3.13.

b. $\mathbb{P}(X > 1, Y < 1) = \frac{5}{18}$

3.14.

b. $\mathbb{P}\left(V < \frac{1}{2}, X = 2\right) = \frac{33}{320} \approx 0.1031$

4. Joint Distributions

4.6. The joint and marginal probability density functions are given in the table below; Y and Z are dependent.

$\mathbb{P}(Y = y, Z = z)$		y											$\mathbb{P}(Z = z)$
		2	3	4	5	6	7	8	9	0	11	12	
z	-5	0	0	0	0	0	$\frac{1}{36}$	0	0	0	0	0	$\frac{1}{36}$
	-4	0	0	0	0	$\frac{1}{36}$	0	$\frac{1}{36}$	0	0	0	0	$\frac{2}{36}$
	-3	0	0	0	$\frac{1}{36}$	0	$\frac{1}{36}$	0	$\frac{1}{36}$	0	0	0	$\frac{3}{36}$
	-2	0	0	$\frac{1}{36}$	0	$\frac{1}{36}$	0	$\frac{1}{36}$	0	$\frac{1}{36}$	0	0	$\frac{4}{36}$
	-1	0	$\frac{1}{36}$	0	$\frac{1}{36}$	0	$\frac{1}{36}$	0	$\frac{1}{36}$	0	$\frac{1}{36}$	0	$\frac{5}{36}$
	0	$\frac{1}{36}$	0	$\frac{1}{36}$	0	$\frac{1}{36}$	0	$\frac{1}{36}$	0	$\frac{1}{36}$	0	$\frac{1}{36}$	$\frac{6}{36}$
	1	0	$\frac{1}{36}$	0	$\frac{1}{36}$	0	$\frac{1}{36}$	0	$\frac{1}{36}$	0	$\frac{1}{36}$	0	$\frac{5}{36}$
	2	0	0	$\frac{1}{36}$	0	$\frac{1}{36}$	0	$\frac{1}{36}$	0	$\frac{1}{36}$	0	0	$\frac{4}{36}$
	3	0	0	0	$\frac{1}{36}$	0	$\frac{1}{36}$	0	$\frac{1}{36}$	0	0	0	$\frac{3}{36}$
	4	0	0	0	0	$\frac{1}{36}$	0	$\frac{1}{36}$	0	0	0	0	$\frac{2}{36}$

	5	0	0	0	0	0	$\frac{1}{36}$	0	0	0	0	0	$\frac{1}{36}$
$\mathbb{P}(Y = y)$		$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	1

4.7. The joint and marginal probability density functions are given below; U and V are dependent.

$\mathbb{P}(U = u,\ V = v)$		u						$\mathbb{P}(V = v)$
		1	2	3	4	5	6	
v	1	$\frac{1}{36}$	0	0	0	0	0	$\frac{1}{36}$
	2	$\frac{2}{36}$	$\frac{1}{36}$	0	0	0	0	$\frac{3}{36}$
	3	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	0	0	0	$\frac{5}{36}$
	4	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	0	0	$\frac{7}{36}$
	5	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	0	$\frac{9}{36}$
	6	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	$\frac{11}{36}$
$\mathbb{P}(U = u)$		$\frac{11}{36}$	$\frac{9}{36}$	$\frac{7}{36}$	$\frac{5}{36}$	$\frac{3}{36}$	$\frac{1}{36}$	1

4.8.

a. $g(x) = x + \frac{1}{2},\ 0 \le x \le 1$
b. $h(y) = y + \frac{1}{2},\ 0 \le y \le 1$
c. X and Y are dependent.

4.9.

a. $g(x) = (1 + 3\,x)\,(1 - x),\ 0 \le x \le 1$
b. $h(y) = 3\,y^2,\ 0 \le y \le 1$
c. X and Y are dependent.

4.10.

a. $g(x) = 3\,x^2,\ 0 \le x \le 1$
b. $h(y) = 2\,y,\ 0 \le y \le 1$
c. X and Y are independent.

4.11.

a. $g(x) = \frac{15}{2}\left(x^2 - x^4\right),\ 0 \le x \le 1$
b. $h(y) = 5\,y^4,\ 0 \le y \le 1$
c. X and Y are dependent.

4.12.

a. $f_{X,Y}\left(x,\ y\right) = x + y,\ 0 \le x \le 1,\ 0 \le y \le 1$
b. $f_{X,Z}\left(x,\ z\right) = 2\,z\left(x + \frac{1}{2}\right),\ 0 \le x \le 1,\ 0 \le z \le 1$
c. $f_{Y,Z}\left(y,\ z\right) = 2\,z\left(y + \frac{1}{2}\right),\ 0 \le y \le 1,\ 0 \le z \le 1$
d. $f_X(x) = x + \frac{1}{2},\ 0 \le x \le 1$
e. $f_Y(y) = y + \frac{1}{2},\ 0 \le y \le 1$

f. $f_Z(z) = 2\,z,\ 0 \le z \le 1$
g. Z and $(X,\ Y)$ are independent; X and Y are dependent.

4.13.

a. $g(x) = 2\,e^{-2\,x},\ x > 0$. X has an exponential distribution.
b. $h(y) = 2\left(e^{-y} - e^{-2\,y}\right),\ y > 0$
c. X and Y are dependent.

4.17.

a. $f(x,\ y) = \frac{1}{144},\ -6 \le x \le 6,\ -6 \le y \le 6$; $g(x) = \frac{1}{12},\ -6 \le x \le 6$; $h(y) = \frac{1}{12},\ -6 \le y \le 6$; X and Y are independent.
b. $f(x,\ y) = \frac{1}{72},\ -6 \le y \le x \le 6$; $g(x) = \frac{1}{72}(x+6),\ -6 \le x \le 6$; $h(y) = \frac{1}{72}(6-y),\ -6 \le y \le 6$; X and Y are dependent.
c. $f(x,\ y) = \frac{1}{36\,\pi},\ x^2 + y^2 \le 36$; $g(x) = \frac{\sqrt{36-x^2}}{18\,\pi},\ -6 \le x \le 6$; $h(y) = \frac{\sqrt{36-y^2}}{18\,\pi},\ -6 \le y \le 6$; X and Y are dependent.

4.19.

a. $f(x,\ y,\ z) = 1,\ 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1$ (the uniform distribution on $[0,\ 1]^3$
b. $(X,\ Y)$, $(X,\ Z)$, and $(Y,\ Z)$ have common probability density function $g(u,\ v) = 1,\ 0 \le u \le 1,\ 0 \le v \le 1$ (the uniform distribution on $[0,\ 1]^2$
c. X, Y, and Z have common probability density function $h(u) = 1,\ 0 \le u \le 1$ (the uniform distribution on $[0,\ 1]$
d. X, Y, Z are (mutually) independent.

4.20. In the notation below, the subscripts refer to the variable number.

a. $f(x,\ y,\ z) = 6,\ 0 \le x \le y \le z \le 1$
b. $f_{1,2}(x,\ y) = 6\,(1-y),\ 0 \le x \le y \le 1$
c. $f_{1,3}(x,\ z) = 6\,(z-x),\ 0 \le x \le z \le 1$
d. $f_{2,3}(y,\ z) = 6\,y,\ 0 \le y \le z \le 1$
e. $f_1(x) = 3\,(1-x)^2,\ 0 \le x \le 1$
f. $f_2(y) = 6\,y\,(1-y),\ 0 \le y \le 1$
g. $f_3(z) = 3\,z^2,\ 0 \le z \le 1$
h. Each pair of variables is dependent.

4.25. In the formulas below, the variables i, j, and k are nonnegative integers.

a. $\mathbb{P}(U = i,\ V = j,\ W = k) = \dfrac{\binom{13}{i}\binom{13}{j}\binom{13}{k}\binom{13}{13-i-j-k}}{\binom{52}{13}},\quad i + j + k \le 13$
b. $\mathbb{P}(U = i,\ V = j) = \dfrac{\binom{13}{i}\binom{13}{j}\binom{26}{13-i-j}}{\binom{52}{13}},\quad i + j \le 13$
c. $\mathbb{P}(U = i) = \dfrac{\binom{13}{i}\binom{39}{13-i}}{\binom{52}{13}},\quad i \le 13$

4.30.

a. $g(x) = \frac{1}{2\sqrt{2\,\pi}}\,e^{-\frac{x^2}{8}},\quad x \in \mathbb{R}$. X also has a normal distribution.
b. $h(y) = \frac{1}{3\sqrt{2\,\pi}}\,e^{-\frac{y^2}{18}},\quad y \in \mathbb{R}$. Y also has a normal distribution.
c. X and Y are independent.

4.31.

a. $g(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad x \in \mathbb{R}$. X has the standard normal distribution.

b. $h(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}, \quad y \in \mathbb{R}$. Y has the standard normal distribution.

c. X and Y are dependent.

4.34.

a. $g(x) = \frac{1}{3}$, $x \in \{1, 2, 3\}$ (the uniform distribution on $\{1, 2, 3\}$).

b. $h(y) = \begin{cases} \frac{11}{18}, & 0 < y < 1 \\ \frac{5}{18}, & 1 < y < 2 \\ \frac{2}{18}, & 2 < y < 3 \end{cases}$

c. X and Y are dependent.

4.35.

a. $g(p) = 6\,p\,(1 - p), 0 \le p \le 1$

b. $h(y) = \begin{cases} \frac{1}{5}, & y = 0 \\ \frac{3}{10}, & y = 1 \\ \frac{1}{5}, & y = 3 \end{cases}$

c. X and Y are dependent.

4.36. The empirical joint and marginal empirical densities are given in the table below. Gender and species are probably dependent (compare the joint density with the product of the marginal densities).

$\mathbb{P}(G = i, S = j)$		$i = 0$	$i = 1$	$\mathbb{P}(S = j)$
j	0	$\frac{16}{104}$	$\frac{28}{104}$	$\frac{44}{104}$
	1	$\frac{3}{104}$	$\frac{3}{104}$	$\frac{6}{104}$
	2	$\frac{40}{104}$	$\frac{14}{104}$	$\frac{56}{104}$
$\mathbb{P}(G = i)$		$\frac{59}{104}$	$\frac{45}{104}$	1

4.37. The empirical joint and marginal densities, based on simple partitions of the body weight and body length ranges, are given in the table below. Body weight and body length are almost certainly dependent.

Density (W_B, L_B)		W_B (0, 0.1]	W_B (0.1, 0.2]	W_B (0.2, 0.3]	W_B (0.3, 0.4]	Density L_B
L_B	(15, 20]	0	0.0385	0.0192	0	0.0058
	(20, 25]	0.1731	0.9808	0.4231	0	0.1577
	(25, 30]	0	0.1538	0.1731	0.0192	0.0346
	(30, 35]	0	0	0	0.0192	0.0019
Density W_B		0.8654	5.8654	3.0769	0.1923	

4.38. The empirical joint and marginal densities, based on a simple partition of the body weight range, are given in the table below.

Body weight and gender are almost certainly dependent.

Density (W_B, G)		W_B (0, 0.1]	(0.1, 0.2]	(0.2, 0.3]	(0.3, 0.4]	Density G
G	0	0.1923	2.5000	2.8846	0.0962	0.5673
	1	0.6731	3.3654	0.1923	0.0962	0.4327
Density W_B		0.8654	5.8654	3.0769	0.1923	

5. Conditional Distributions

5.9. The conditional probability density functions of U given the different values of V are recorded in the table below.

$\mathbb{P}(U = u \mid V = v)$		u 1	2	3	4	5	6
v	1	1	0	0	0	0	0
	2	$\frac{2}{3}$	$\frac{1}{3}$	0	0	0	0
	3	$\frac{2}{5}$	$\frac{2}{5}$	$\frac{1}{5}$	0	0	0
	4	$\frac{2}{7}$	$\frac{2}{7}$	$\frac{2}{7}$	$\frac{1}{7}$	0	0
	5	$\frac{2}{9}$	$\frac{2}{9}$	$\frac{2}{9}$	$\frac{2}{9}$	$\frac{1}{9}$	0
	6	$\frac{2}{11}$	$\frac{2}{11}$	$\frac{2}{11}$	$\frac{2}{11}$	$\frac{2}{11}$	$\frac{1}{11}$

5.10. The joint and marginal probability density functions are given in the first table. The conditional probability density function of N given the different values of X are recorded in the second table.

$\mathbb{P}(N = n, Y = k)$		n 1	2	3	4	5	6	$\mathbb{P}(Y = k)$
k	0	$\frac{1}{12}$	$\frac{1}{24}$	$\frac{1}{48}$	$\frac{1}{96}$	$\frac{1}{192}$	$\frac{1}{384}$	$\frac{63}{384}$
	1	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{16}$	$\frac{1}{24}$	$\frac{5}{192}$	$\frac{1}{64}$	$\frac{120}{384}$
	2	0	$\frac{1}{24}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{5}{96}$	$\frac{5}{128}$	$\frac{99}{384}$
	3	0	0	$\frac{1}{48}$	$\frac{1}{24}$	$\frac{5}{96}$	$\frac{5}{96}$	$\frac{64}{384}$
	4	0	0	0	$\frac{1}{96}$	$\frac{5}{192}$	$\frac{5}{128}$	$\frac{29}{384}$
	5	0	0	0	0	$\frac{1}{192}$	$\frac{1}{64}$	$\frac{8}{384}$
	6	0	0	0	0	0	$\frac{1}{384}$	$\frac{1}{384}$
$\mathbb{P}(N = n)$		$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	1

$\mathbb{P}(N = n \mid Y = k)$		1	2	3	4	5	6
k	0	$\frac{32}{63}$	$\frac{16}{63}$	$\frac{8}{63}$	$\frac{4}{63}$	$\frac{2}{63}$	$\frac{1}{63}$
	1	$\frac{16}{60}$	$\frac{16}{60}$	$\frac{12}{60}$	$\frac{8}{60}$	$\frac{5}{60}$	$\frac{3}{60}$
	2	0	$\frac{16}{99}$	$\frac{24}{99}$	$\frac{24}{99}$	$\frac{20}{99}$	$\frac{15}{99}$
	3	0	0	$\frac{2}{16}$	$\frac{4}{16}$	$\frac{5}{16}$	$\frac{5}{16}$

	4	0	0	0	$\frac{4}{29}$	$\frac{10}{29}$	$\frac{15}{29}$
	5	0	0	0	0	$\frac{1}{4}$	$\frac{3}{14}$
	6	0	0	0	0	0	1

5.12. The joint and marginal probability density functions are given in the first table below. The conditional probability density functions of X given the different values of Y are recorded in the second table.

$\mathbb{P}(X = i, Y = k)$		k						$\mathbb{P}(X = i)$
		1	2	3	4	5	6	
i	0	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{2}$
	1	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{2}$
$\mathbb{P}(Y = k)$		$\frac{5}{24}$	$\frac{7}{48}$	$\frac{7}{48}$	$\frac{7}{48}$	$\frac{7}{48}$	$\frac{5}{24}$	1

$\mathbb{P}(X = i \mid Y = k)$		k					
		1	2	3	4	5	6
i	0	$\frac{2}{5}$	$\frac{4}{7}$	$\frac{4}{7}$	$\frac{4}{7}$	$\frac{4}{7}$	$\frac{2}{5}$
	1	$\frac{3}{5}$	$\frac{3}{7}$	$\frac{3}{7}$	$\frac{3}{7}$	$\frac{3}{7}$	$\frac{3}{5}$

5.14. The joint probability density function of (V, Y) and the marginal probability density function of Y are given in the first table. The conditional probability density function of V given the different values of Y are recorded in the second table.

$\mathbb{P}(V = p, Y = k)$		k			$\mathbb{P}(V = p)$
		0	1	2	
p	$\frac{1}{2}$	$\frac{5}{48}$	$\frac{10}{48}$	$\frac{5}{48}$	$\frac{5}{12}$
	$\frac{1}{3}$	$\frac{4}{27}$	$\frac{4}{27}$	$\frac{1}{27}$	$\frac{4}{12}$
	1	0	0	$\frac{1}{4}$	$\frac{3}{12}$
$\mathbb{P}(Y = k)$		$\frac{109}{432}$	$\frac{154}{432}$	$\frac{169}{432}$	1

$\mathbb{P}(V = p \mid Y = k)$		k		
		0	1	2
p	$\frac{1}{2}$	$\frac{45}{109}$	$\frac{45}{77}$	$\frac{45}{169}$
	$\frac{1}{3}$	$\frac{64}{109}$	$\frac{32}{77}$	$\frac{16}{169}$
	1	0	0	$\frac{108}{169}$

5.15.

a. $f(p, k) = 6\binom{3}{k} p^{k+1} (1-p)^{4-k}$ $0 \le p \le 1, k \in \{0, 1, 2, 3\}$

b. $h(k) = \begin{cases} \frac{1}{5}, & k = 0 \\ \frac{3}{10}, & k = 1 \\ \frac{3}{10}, & k = 2 \\ \frac{1}{5}, & k = 3 \end{cases}$

c. $g(p|k) = \begin{cases} 30\, p\, (1-p)^4, & k = 0 \\ 60\, p^2\, (1-p)^3, & k = 1 \\ 60\, p^3\, (1-p)^2, & k = 2 \\ 30\, p^4\, (1-p), & k = 3 \end{cases}$

5.16. Let N denote the bulb number and T the lifetime.

a. $\mathbb{P}(T > 1) = 0.1156$

b.

n	1	2	3	4	5
$\mathbb{P}(N = n \mid T > 1)$	0.6364	0.2341	0.0861	0.0317	0.0117

5.17.

a. $\mathbb{P}(N = n,\ Y = k) = e^{-1}\, \frac{p^k}{k!}\, \frac{(1-p)^{n-k}}{(n-k)!}$, $n \in \mathbb{N}$, $k \in \{0,\ 1,\ \ldots,\ n\}$

b. $\mathbb{P}(Y = k) = e^{-p}\, \frac{p^k}{k!}$, $k \in \mathbb{N}$ (Poisson with parameter p)

c. $\mathbb{P}(N = n \mid X = k) = e^{-(1-p)}\, \frac{(1-p)^{n-k}}{(n-k)!}$, $n \in \{k,\ k+1,\ \ldots\}$

5.18.

a. $f(i,\ y) = \frac{1}{3\, i}$, $i \in \{1,\ 2,\ 3\}$, $0 \le y \le i$

b. $h(y) = \begin{cases} \frac{11}{18}, & 0 < y < 1 \\ \frac{5}{18}, & 1 < y < 2 \\ \frac{2}{18}, & 2 < y < 3 \end{cases}$

c. $g(i|y) = \begin{cases} \frac{6}{11}, & i = 1,\ 0 < y < 1 \\ \frac{3}{11}, & i = 2,\ 0 < y < 1 \\ \frac{2}{11}, & i = 3,\ 0 < y < 1 \\ 0, & i = 1,\ 1 < y < 2 \\ \frac{3}{5}, & i = 2,\ 1 < y < 2 \\ \frac{2}{5}, & i = 3,\ 1 < y < 2 \\ 0, & i = 1,\ 2 < y < 3 \\ 0, & i = 2,\ 2 < y < 3 \\ 1, & i = 3,\ 2 < y < 3 \end{cases}$

5.19.

a. $g(x|y) = \frac{x+y}{y+\frac{1}{2}}$, $0 < x < 1$, $0 < y < 1$
b. $h(y|x) = \frac{x+y}{x+\frac{1}{2}}$, $0 < x < 1$, $0 < y < 1$
c. X and Y are dependent.

5.20.

a. $g(x|y) = \frac{x+y}{3\,y^2}$, $0 < x < y < 1$
b. $h(y|x) = \frac{x+y}{(1+3\,x)\,(1-x)}$, $0 < x < y < 1$
c. X and Y are dependent.

5.21.

a. $g(x|y) = \frac{3\,x^2}{y^3}$, $0 < x < y < 1$
b. $h(y|x) = \frac{2\,y}{1-x^2}$, $0 < x < y < 1$
c. X and Y are dependent.

5.22.

a. $g(x|y) = 3\,x^2$, $0 < x < 1$, $0 < y < 1$
b. $h(y|x) = 2\,y$, $0 < x < 1$, $0 < y < 1$
c. X and Y are independent.

5.23.

a. $g(x|y) = \frac{e^{-x}}{1-e^{-y}}$, $0 < x < y < \infty$
b. $h(y|x) = e^{x-y}$, $0 < x < y < \infty$
c. X and Y are dependent.

5.24.

a. $f(x,\ y) = \frac{1}{x}$, $0 < y < x < 1$
b. $h(y) = -\ln(y)$, $0 < y < 1$
c. $g(x|y) = -\frac{1}{x\ln(y)}$, $0 < y < x < 1$

5.26.

a. $h(y|x) = \frac{1}{12}$, $-6 < x < 6$, $-6 < y < 6$; $g(x|y) = \frac{1}{12}$, $-6 < x < 6$, $-6 < y < 6$; X , Y are independent.
b. $h(y|x) = \frac{1}{x+6}$, $-6 < y < x < 6$; $g(x|y) = \frac{1}{6-y}$, $-6 < y < x < 6$; X, Y are dependent.
c. $h(y|x) = \frac{1}{2\,\sqrt{36-x^2}}$, $x^2 + y^2 < 36$; $g(x|y) = \frac{1}{2\,\sqrt{36-y^2}}$, $x^2 - y^2 < 36$; X, Y are dependent.

5.28. In the notation below, the subscripts refer to the variable numbers

a. $f_{1,2|3}\left(x,\ y|z\right) = \frac{2}{z^2}$, $0 < x < y < z < 1$
b. $f_{1,3|2}\left(x,\ z|y\right) = \frac{1}{y\,(1-y)}$, $0 < x < y < z < 1$
c. $f_{2,3|1}\left(y,\ z|x\right) = \frac{2}{(1-x)^2}$, $0 < x < y < z < 1$
d. $f_{1|2,3}\left(x|y,\ z\right) = \frac{1}{y}$, $0 < x < y < z < 1$
e. $f_{2|1,3}\left(y|x,\ z\right) = \frac{1}{z-x}$, $0 < x < y < z < 1$

f. $f_{3|1,2}(z|x, y) = \frac{1}{1-y}$, $0 < x < y < z < 1$

5.31. In the formulas below, the variables i and j are nonnegative integers.

a. $\mathbb{P}(U = i, V = j | W = 3) = \frac{\binom{13}{i}\binom{13}{j}\binom{13}{10-i-j}}{\binom{39}{10}}, \quad i + j \le 10$

b. $\mathbb{P}(U = i | V = 3, W = 2) = \frac{\binom{13}{i}\binom{13}{8-i}}{\binom{26}{8}}, \quad i \le 8$

5.35.

a. $g(x|y) = \frac{1}{2\sqrt{2\pi}} e^{-\frac{x^2}{8}}, \quad x \in \mathbb{R}, y \in \mathbb{R}.$

b. $h(y|x) = \frac{1}{3\sqrt{2\pi}} e^{-\frac{y^2}{18}}, \quad y \in \mathbb{R}, x \in \mathbb{R}.$

c. X and Y are independent.

5.36. X and Y have the same distribution.

a. $g(x|y) = \sqrt{\frac{2}{3\pi}} e^{-\frac{2}{3}\left(x - \frac{1}{2}y\right)^2}, \quad x \in \mathbb{R}, y \in \mathbb{R}$

b. $h(y|x) = \sqrt{\frac{2}{3\pi}} e^{-\frac{2}{3}\left(y - \frac{1}{2}x\right)^2}, \quad y \in \mathbb{R}, x \in \mathbb{R}$

6. Distribution Functions

6.26.

b. $f(x) = \begin{cases} \frac{1}{10}, & x = 1 \\ \frac{1}{5}, & x = \frac{3}{2} \\ \frac{3}{10}, & x = 2 \\ \frac{3}{10}, & x = \frac{5}{2} \\ \frac{1}{10}, & x = 3 \end{cases}$

c. $\mathbb{P}(2 \le X < 3) = \frac{3}{5}$

d. $F^{-1}(p) = \begin{cases} 1, & 0 < p \le \frac{1}{10} \\ \frac{3}{2}, & \frac{1}{10} < p \le \frac{3}{10} \\ 2, & \frac{3}{10} < p \le \frac{6}{10} \\ \frac{5}{2}, & \frac{6}{10} < p \le \frac{9}{10} \\ 3, & \frac{9}{10} < p \le 1 \end{cases}$

e. $\left(1, \frac{3}{2}, 2, \frac{5}{2}, 3\right)$

6.27.

b. $f(x) = \frac{1}{(x+1)^2}$, $x > 0$

c. $\mathbb{P}(2 \le X < 3) = \frac{1}{12}$

d. $F^{-1}(p) = \frac{p}{1-p}$, $0 < p < 1$
e. $\left(0, \frac{1}{3}, 1, 3, \infty\right)$

6.28.

b. $g(1) = g(2) = g(3) = \frac{1}{12}$
c. $h(x) = \begin{cases} \frac{1}{4}, & 0 < x < 1 \\ \frac{1}{2}(x-1), & 1 < x < 2 \\ \frac{3}{4}(x-1)^2, & 2 < x < 3 \end{cases}$
d. $\mathbb{P}(2 \le X < 3) = \frac{1}{3}$
e. $F^{-1}(p) = \begin{cases} 4p, & 0 < p \le \frac{1}{4} \\ 1, & \frac{1}{4} < p \le \frac{1}{3} \\ 1 + \sqrt{4\left(p - \frac{1}{3}\right)}, & \frac{1}{3} < p \le \frac{7}{12} \\ 2, & \frac{7}{12} < p \le \frac{2}{3} \\ 2 + \sqrt[3]{4\left(p - \frac{2}{3}\right)}, & \frac{2}{3} < p \le \frac{11}{12} \\ 3, & \frac{11}{12} < p \le 1 \end{cases}$
f. $\left(0, 1, 1 + \sqrt{\frac{2}{3}}, 2 + \sqrt[3]{\frac{1}{3}}, 3\right)$

6.29.

b. $F(x) = \frac{x-a}{b-a}$, $a \le x < b$
c. $F^{-1}(p) = a + (b-a)p$, $0 < p < 1$
d. $\left(a, \frac{3a+b}{4}, \frac{a+b}{2}, \frac{a+3b}{4}, b\right)$

6.30.

b. $F(t) = 1 - e^{-rt}$, $t \ge 0$
c. $G(t) = e^{-rt}$, $t \ge 0$
d. $h(t) = r$, $t \ge 0$
e. $F^{-1}(p) = \frac{-\ln(1-p)}{r}$, $0 < p < 1$
f. $\left(0, \frac{\ln(4)-\ln(3)}{r}, \frac{\ln(2)}{r}, \frac{\ln(4)}{r}, \infty\right)$

6.32.

b. $F(x) = 1 - \frac{1}{x^a}$, $x \ge 1$
c. $G(x) = \frac{1}{x^a}$, $x \ge 1$
d. $h(x) = \frac{a}{x}$, $x \ge 1$
e. $F^{-1}(p) = (1-p)^{-1/a}$, $0 < p < 1$
f. $\left(1, \left(\frac{3}{4}\right)^{-1/a}, \left(\frac{1}{2}\right)^{-1/a}, \left(\frac{1}{4}\right)^{-1/a}, \infty\right)$

6.34.

a. $F(x) = \frac{1}{2} + \frac{1}{\pi} \arctan(x)$, $x \in \mathbb{R}$
b. $F^{-1}(p) = \tan\left(\pi \left(p - \frac{1}{2}\right)\right)$, $0 < p < 1$
c. $(-\infty, -1, 0, 1, \infty)$, $\text{IQR} = 2$

6.36.

c. $G(t) = \exp\left(-t^k\right)$, $t > 0$
d. $F(t) = 1 - \exp\left(-t^k\right)$, $t > 0$
e. $f(t) = k\, t^{k-1} \exp\left(-t^k\right)$, $t > 0$
f. $F^{-1}(p) = (-\ln(1-p))^{1/k}$, $0 < p < 1$
g. $\left(0, (\ln(4) - \ln(3))^{1/k}, \ln(2)^{1/k}, \ln(4)^{1/k}, \infty\right)$

6.38.

a. $F(x) = 4\,x^3 - 3\,x^4$, $0 \le x \le 1$
b. $(0, 0.4563, 0.6413, 0.7570, 1)$, $\text{IQR} = 0.3007$

6.39.

a. $F(x) = \frac{2}{\pi} \arcsin\left(\sqrt{x}\right)$, $0 \le x \le 1$
b. $F^{-1}(p) = \sin\left(\frac{\pi}{2}\, p\right)^2$, $0 < p < 1$
c. $\left(0, \frac{1}{2} - \frac{\sqrt{2}}{4}, \frac{1}{2}, \frac{1}{2} + \frac{\sqrt{2}}{4}, 1\right)$, $\text{IQR} = \frac{\sqrt{2}}{2}$

6.41.

b. $F^{-1}(p) = \ln\left(\frac{p}{1-p}\right)$, $0 < p < 1$
c. $(-\infty, -\ln(3), 0, \ln(3), \infty)$
d. $f(x) = \frac{e^x}{(1+e^x)^2}$, $x \in \mathbb{R}$

6.43.

b. $F^{-1}(p) = -\ln(-\ln(p))$, $0 < p < 1$
c. $(-\infty, -\ln(\ln(4)), -\ln(\ln(2)), -\ln(\ln(4) - \ln(3)), \infty)$
d. $f(x) = e^{-e^{-x}}\, e^{-x}$, $x \in \mathbb{R}$

6.45.

b. $F(x) = x - x\ln(x)$, $0 < x < 1$
c. $\mathbb{P}\left(\frac{1}{3} \le X \le \frac{1}{2}\right) = \frac{1}{6} + \frac{1}{2}\ln(2) - \frac{1}{3}\ln(3)$

6.47.

a.

y	$(-\infty, 2)$	$[2, 3)$	$[3, 4)$	$[4, 5)$	$[5, 6)$	$[6, 7)$	$[7, 8)$	$[8, 9)$	$[9, 10)$	$[10, 11)$	$[11, 12)$	$[12, \infty)$
$\mathbb{P}(Y \le y)$	0	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{6}{36}$	$\frac{10}{36}$	$\frac{15}{36}$	$\frac{21}{36}$	$\frac{26}{36}$	$\frac{30}{36}$	$\frac{33}{36}$	$\frac{35}{36}$	1

b.

v	$(-\infty, 1)$	$[1, 2)$	$[2, 3)$	$[3, 4)$	$[4, 5)$	$[5, 6)$	$[6, \infty)$
$\mathbb{P}(V \le v)$	0	$\frac{1}{36}$	$\frac{4}{36}$	$\frac{9}{36}$	$\frac{16}{36}$	$\frac{25}{36}$	1

c.

y	$(-\infty, 6)$	$[6, 7)$	$[7, 8)$	$[8, 9)$	$[9, 10)$	$[10, \infty)$
$\mathbb{P}(Y \le y \mid V = 5)$	0	$\frac{2}{9}$	$\frac{4}{9}$	$\frac{6}{9}$	$\frac{8}{9}$	1

6.48.

a. $F(x, y) = \frac{1}{2}\left(x y^2 + y x^2\right), 0 < x < 1, 0 < y < 1$
b. $G(x) = \frac{1}{2}\left(x + x^2\right), 0 < x < 1$
c. $H(y) = \frac{1}{2}\left(y + y^2\right), 0 < y < 1$
d. $G(x \mid y) = \frac{\frac{1}{2}x^2 + x y}{y + \frac{1}{2}}, 0 < x < 1, 0 < y < 1$
e. $H(y \mid x) = \frac{\frac{1}{2}y^2 + x y}{x + \frac{1}{2}}, 0 < x < 1, 0 < y < 1$

6.49. Let N denote the total number of candies. The empirical distribution function of N is a step function; the following table gives the values of the function at the jump points.

n	50	53	54	55	56	57	58	59	60	61
$\mathbb{P}(N \le n)$	$\frac{1}{30}$	$\frac{2}{30}$	$\frac{3}{30}$	$\frac{7}{30}$	$\frac{11}{30}$	$\frac{14}{30}$	$\frac{23}{30}$	$\frac{36}{30}$	$\frac{28}{30}$	1

7. Transformations of Variables

7.20. Let $Y = X_1 + X_2$ denote the sum of the scores.

a.

y	2	3	4	5	6	7	8	9	10	11	12
$\mathbb{P}(Y = y)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

b.

y	2	3	4	5	6	7	8	9	10	11	12
$\mathbb{P}(Y = y)$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{5}{64}$	$\frac{3}{32}$	$\frac{7}{64}$	$\frac{3}{16}$	$\frac{7}{64}$	$\frac{3}{32}$	$\frac{6}{64}$	$\frac{1}{16}$	$\frac{1}{16}$

7.22. Let $Y = X_1 + X_2$ denote the sum of the scores.

y	2	3	4	5	6	7	8	9	10	11	12
$\mathbb{P}(Y = y)$	$\frac{2}{48}$	$\frac{3}{48}$	$\frac{4}{48}$	$\frac{5}{48}$	$\frac{6}{48}$	$\frac{8}{48}$	$\frac{6}{48}$	$\frac{5}{48}$	$\frac{4}{48}$	$\frac{3}{48}$	$\frac{2}{48}$

7.23. Let U denote the minimum score and V the maximum score.

a. $\mathbb{P}(U = k) = \left(1 - \frac{k-1}{6}\right)^n - \left(1 - \frac{k}{6}\right)^n, k \in \{1, 2, 3, 4, 5, 6\}$
b. $\mathbb{P}(V = k) = \left(\frac{k}{6}\right)^n - \left(\frac{k-1}{6}\right)^n, k \in \{1, 2, 3, 4, 5, 6\}$

7.25.

a. $g(y) = \frac{1}{4\sqrt{y}}, 0 < y < 1$
b. $g(y) = \begin{cases} \frac{1}{4\sqrt{y}}, & 0 < y < 1 \\ \frac{1}{8\sqrt{y}}, & 1 < y < 9 \end{cases}$

c. $g(y) = \frac{1}{4\sqrt{y}}, \quad 4 < y < 16$

7.27.

b. $g(u, v) = \frac{1}{2}$ for (u, v) in the square with vertices $\{(0, 0), (1, 1), (2, 0), (1, -1)\}$. Thus (U, V) is uniformly distributed on this square.

c. $h(u) = \begin{cases} u, & 0 < u < 1 \\ 2 - u, & 1 < u < 2 \end{cases}$

d. $k(v) = \begin{cases} 1 - v, & 0 < v < 1 \\ 1 + v, & -1 < v < 0 \end{cases}$

7.28. $g(u, v, w) = \frac{1}{2}$ for (u, v, w) in the rectangular region of $\mathbb{R}^3$ with vertices $\{(0, 0, 0), (1, 0, 1), (1, 1, 0), (0, 1, 1), (2, 1, 1), (1, 1, 2), (1, 2, 1), (2, 2, 2)\}$

7.29.

a. $G(t) = 1 - (1 - t)^n$, $0 < t < 1$ and $g(t) = n(1 - t)^{n-1}$, $0 < t < 1$

b. $H(t) = t^n$, $0 < t < 1$ and $h(t) = n\,t^{n-1}$, $0 < t < 1$

7.31.

a. $f^{*2}(z) = \begin{cases} z, & 0 < z < 1 \\ 2 - z, & 1 < z < 2 \end{cases}$

b. $f^{*3}(z) = \begin{cases} \frac{1}{2} z^2, & 0 < z < 1 \\ 1 - \frac{1}{2}(z - 1)^2 - \frac{1}{2}(2 - z)^2, & 1 < z < 2 \\ \frac{1}{2}(3 - z)^2, & 2 < z < 3 \end{cases}$

7.34. $X = a + U(b - a)$ where U is a random number (uniformly distributed on $(0, 1)$).

7.38. $f(k) = \binom{n}{(n+k)/2} p^{(n+k)/2} (1 - p)^{(n-k)/2}$ for $k \in \{-n, 2 - n, ..., n - 2, n\}$.

7.40. $X = \frac{-\ln(1 - U)}{r}$ where U is a random number (uniformly distributed on $(0, 1)$).

7.41. Let $Y = \lfloor T \rfloor$ and $Z = \lceil T \rceil$.

a. $\mathbb{P}(Y = n) = e^{-r\,n}(1 - e^{-r})$, $n \in \mathbb{N}$

b. $\mathbb{P}(Z = n) = e^{-r\,(n-1)}(1 - e^{-r})$, $n \in \mathbb{N}_+$

7.42.

a. $G(z) = 1 - \frac{1}{1+z}$, $z > 0$

b. $g(z) = \frac{1}{(1+z)^2}$, $z > 0$

7.43. Let h denote the probability density function of Z.

a. $h(z) = a^2\,z\,e^{-a\,z}$, $z > 0$ if $a = b$

b. $h(z) = \frac{a\,b}{b-a}\left(e^{-a\,z} - e^{-b\,z}\right)$, $z > 0$ if $a \neq b$

7.44.

c. $G(t) = e^{-n\,r\,t}$, $t > 0$ and $g(t) = n\,r\,e^{-n\,r\,t}$, $t > 0$

d. $H(t) = 1 - (1 - e^{-r\,t})^n$, $t > 0$ and $h(t) = n\,r\,e^{-r\,t}\,(1 - e^{-r\,t})^{n-1}$, $t > 0$

7.49.

a. $G(y) = 1 - e^{-a\,y}$, $y > 0$
b. $g(y) = a\,e^{-a\,y}$, $y > 0$

7.48. $h(x) = \frac{1}{(n-1)!}\,e^{-e^x}\,e^{n\,x}$ for $x \in \mathbb{R}$

7.50. $g(y) = a\,y^{a-1}, \quad 0 < y < 1$

7.51. $X = \frac{1}{(1-U)^{1/a}}$ where U is a random number (uniformly distributed on $(0,\ 1)$).

7.52. $f(x) = \frac{1}{\sqrt{2\,\pi}\,\sigma}\,e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ for $x \in \mathbb{R}$.

7.53. $g(v) = \frac{1}{\sqrt{2\,\pi\,v}}\,e^{-\frac{1}{2}\,v}$ for $v \in (0,\ \infty)$.

7.58. $g(y) = 9\,y^8$ for $0 \le y \le 1$.

8. Convergence in Distribution

8.16.

a. Point mass at $x = 1$
b. The standard exponential distribution, with distribution function $G(y) = 1 - e^{-y}$ for $0 < y < \infty$
