3. Random Triangles

Preliminaries

Statement of the Problem

Suppose that a stick is randomly broken in two places. What is the probability that the three pieces form a triangle?

1. Without looking below, make a guess.

2. Run the triangle experiment 50 times. Do not be concerned with all of the information displayed in the applet, but just note whether the pieces form a triangle. Would you like to revise you guess in Exercise 1?

Mathematical Formulation

As usual, the first step is to model the random experiment mathematically. We will take the length of the stick as our unit of length, so that we can identify the stick with the interval $[0, 1]$. To break the stick into three pieces, we just need to select two points in the interval. Thus, let X denote the first point chosen and Y the second point chosen. Note that X and Y are random variables and hence the sample space of our experiment is

$$S = [0, 1]^2$$

Now, to model the statement that the points are chosen *at random*, let us assume, as in the previous sections, that X and Y are independent and each is uniformly distributed on $[0, 1]$.

3. Show that (X, Y) is uniformly distributed on $S = [0, 1]^2$.

Hence

$$\mathbb{P}((X, Y) \in A) = \frac{\text{area}(A)}{\text{area}(S)}, \quad A \subseteq S$$

The Probability of a Triangle

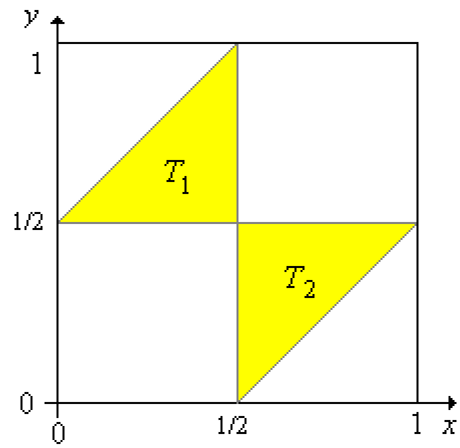
4. Argue that the three pieces form a triangle if and only if the **triangle inequalities** hold: the sum of the lengths of any two pieces must be greater than the length of the third piece.

5. Show that the event that the pieces form a triangle is $T = T_1 \cup T_2$ where

a. $T_1 = \left\{(x, y) \in S : \left(y > \frac{1}{2}\right) \text{ and } \left(x < \frac{1}{2}\right) \text{ and } \left(y - x < \frac{1}{2}\right)\right\}$

b. $T_2 = \left\{(x, y) \in S : \left(x > \frac{1}{2}\right) \text{ and } \left(y < \frac{1}{2}\right) \text{ and } \left(x - y < \frac{1}{2}\right)\right\}$

A sketch of the event T is given below. Curiously, T is composed of triangles!



6. Show that $\mathbb{P}(T) = \frac{1}{4}$.

How close did you come with your guess in Exercise 1? The relative low value of the probability in Exercise 6 is a bit surprising.

7. Run the triangle experiment 1000 times, updating every 10 runs. Note the apparent convergence of the empirical probability of T^c to the true probability.

Triangles of Different Types

Now let us compute the probability that the pieces form a triangle of a given type. Recall that in an **acute** triangle all three angles are less than 90°, while an **obtuse** triangle has one angle (and only one) that is greater than 90°. A **right** triangle, of course, has one 90° angle.

8. Suppose that a triangle has side lengths a, b, and c, where c is the largest of these. Recall (or show) that the triangle is

a. acute if and only if $c^2 < a^2 + b^2$.
b. obtuse if and only if $c^2 > a^2 + b^2$.
c. right if and only if $c^2 = a^2 + b^2$.

Part (c), of course, is the famous Pythagorean theorem, named for the ancient Greek mathematician Pythagoras.

9. Show that the right triangle equations for the stick pieces are

a. $(y - x)^2 = x^2 + (1 - y)^2$ in T_1

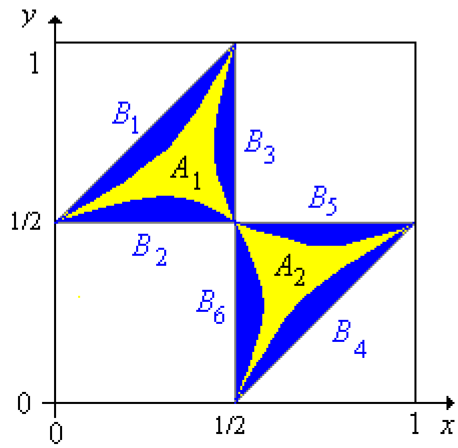
b. $(1-x)^2 = x^2 + (y-x)^2$ in T_1

c. $x^2 = (y-x)^2 + (1-y)^2$ in T_1

d. $(x-y)^2 = y^2 + (1-x)^2$ in T_2

e. $(1-x)^2 = y^2 + (x-y)^2$ in T_2

f. $y^2 = (x-y)^2 + (1-x)^2$ in T_2



10. Let R denote the event that the pieces form a right triangle. Show that $\mathbb{P}(R) = 0$

11. Show that the event that the pieces form an acute triangle is $A = A_1 \cup A_2$ where

a. A_1 is the region inside curves (a), (b), and (c) of Exercise 9.

b. A_1 is the region inside curves (d), (e), and (f) of Exercise 9.

12. Show that the event that the pieces form an obtuse triangle is $B = B_1 \cup B_2 \cup B_3 \cup B_4 \cup B_5 \cup B_6$ where

a. B_1, B_2, and B_3 are the regions inside T_1 and outside of curves (a), (b), and (c) of Exercise 9, respectively.

b. B_4, B_5, and B_6 are the regions inside T_2 and outside of curves (d), (e), and (f) of Exercise 9, respectively.

13. Show that

a. $\mathbb{P}(B_1) = \int_0^{\frac{1}{2}} \frac{x(1-2x)}{2-2x} dx = \frac{3}{8} - \frac{\ln(2)}{2}$

b. $\mathbb{P}(B_2) = \int_0^{\frac{1}{2}} \frac{x(1-2x)}{2-2x} dx = \frac{3}{8} - \frac{\ln(2)}{2}$

c. $\mathbb{P}(B_3) = \int_0^{\frac{1}{2}} \left(y + \frac{1}{2y} - \frac{3}{2}\right) dy = \frac{3}{8} - \frac{\ln(2)}{2}$

14. Argue from symmetry that

$$\mathbb{P}(B) = \frac{9}{4} - 3\ln(2) \approx 0.1706$$

You might also argue from symmetry that $\mathbb{P}(B_i)$ must be the same for each i, even though B_1 and B_2 (for example) are not congruent.

15. Show that

$$\mathbb{P}(A) = 3\ln(2) - 2 \approx 0.07944$$

16. Run the triangle experiment 1000 times, updating every 10 runs. Note the apparent convergence of the empirical probabilities to the true probabilities.
