2. Bertrand's Paradox

Statement of the Problem

Bertrand's problem is to find the probability that a "random chord" on a circle will be longer than the length of a side of the inscribed equilateral triangle. The problem is named after the French mathematician Joseph Louis Bertrand, who studied the problem in 1889.

It turns out, as we will see, that there are (at least) three answers to Bertrand's problem, depending on how one interprets the phrase "random chord". The lack of a unique answer was considered a paradox at the time, because it was assumed (naively, in hindsight) that there should be a single *natural* answer.

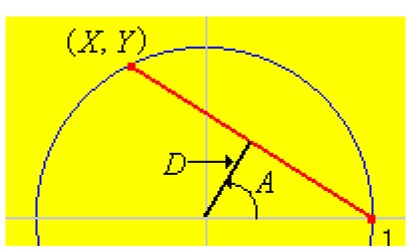
1. Run Bertrand's experiment 100 times, updating after each run, for each of the following models. Do not be concerned with the exact meaning of the models, but see if you can detect a difference in the behavior of the outcomes

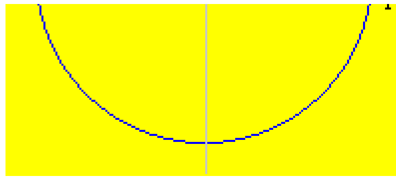
a. Uniform distance
b. Uniform angle
c. Uniform endpoint

Mathematical Formulation

To formulate the problem mathematically, let us take $(0, 0)$ as the center of the circle and take the radius of the circle to be 1. These assumptions entail no loss of generality because they amount to measuring distances relative to the center of the circle, and taking the radius of the circle as the unit of length. Now consider a chord on the circle. By rotating the circle, we can assume that one point of the chord is $(1, 0)$ and the other point is (X, Y) where $Y > 0$ Then we can completely specify the chord by giving any of the following quantities:

- The (perpendicular) distance D from the center of the circle to the midpoint of the chord. Note that $0 \le D \le 1$.
- The angle A between the x-axis and the line from the center of the circle to the midpoint of the chord. Note that $0 \le A \le \frac{\pi}{2}$.
- The horizontal coordinate X. Note that $-1 \le X \le 1$.





2. Show that $D = \cos(A)$.

3. Show that $X = 2\,D^2 - 1$.

4. Show that $Y = 2\,D\,\sqrt{1 - D^2}$

5. Show that the relations in Exercises 2 and 3 are invertible and find the inverse relations.

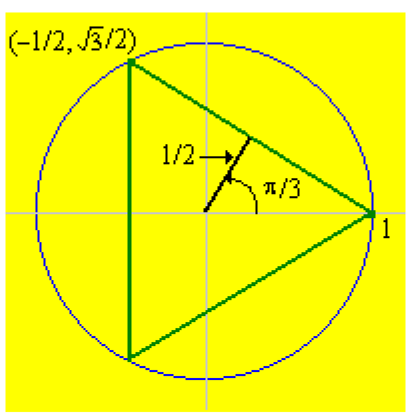
If the chord is generated in a probabilistic way, D, A, X, and Y become random variables. In light of Exercise 5, specifying the distribution of any of the variables D, A, or X completely determines the distribution of all four variables.

6. Show that A is also the angle between the chord and the tangent line to the circle at $(1, 0)$.

Now consider the equilateral triangle inscribed in the circle so that one of the vertices is $(1, 0)$. Consider the chord defined by the upper side of the triangle.

7. Show that for this chord, the angle, distance, and coordinate variables are given as follows:

a. $a = \frac{\pi}{3}$
b. $d = \frac{1}{2}$
c. $x = -\frac{1}{2}$
d. $y = \sqrt{\frac{3}{4}}$



Now suppose that a chord is chosen in probabilistic way.

8. Using Exercise 7, show that the length of the chord is greater than the length of a side of the inscribed equilateral triangle if and only if the following equivalent conditions occur:

a. $0 < D < \frac{1}{2}$
b. $\frac{\pi}{3} < A < \frac{\pi}{2}$
c. $-1 < X < -\frac{1}{2}$

Models

When an object is generated “at random”, a sequence of “natural” variables that determines the object should be given an appropriate uniform distribution. The coordinates of the coin center are such a sequence in Buffon's coin experiment; the angle and distance variables are such a sequence in Buffon's needle experiment. The crux of Bertrand's paradox is the fact that the distance D, the angle A, and the coordinate X each seems to be a natural variable that determine the chord, but different models are obtained, depending on which is given the uniform distribution.

The Model with Uniform Distance

Suppose that D is uniformly distributed on the interval $(0, 1)$.

9. Show that the solution of Bertrand's problem is

$$\mathbb{P}\left(D < \frac{1}{2}\right) = \frac{1}{2}$$

10. In Bertrand's experiment, select the uniform distance model. Run the experiment 1000 times, updating every 10 runs. Note the apparent convergence of the relative frequency function of the chord event to the true probability.

11. Use the change of variables formula to show that the angle A has density function

$$g(a) = \sin(a), \quad 0 < a < \frac{\pi}{2}$$

12. Use the change of variables formula to show that X has density function

$$h(x) = \frac{1}{\sqrt{8\,(x+1)}}, \quad -1 < x < 1$$

Note that A and X are not uniformly distributed.

13. Show how to simulate D, A, X, and Y using a random number.

The Model with Uniform Angle

Suppose that A is uniformly distributed on the interval $\left(0, \frac{\pi}{2}\right)$.

14. Show that the solution of Bertrand's problem is

$$\mathbb{P}\left(A > \frac{\pi}{3}\right) = \frac{1}{3}$$

15. In Bertrand's experiment, select the uniform angle model. Run the experiment 1000 times, updating every 10 runs. Note the apparent convergence of the relative frequency function of the chord event to the true probability.

16. Use the change of variables formula to show that the distance D has density function

$$f(d) = \frac{2}{\pi \sqrt{1 - d^2}}, \quad 0 < d < 1$$

17. Use the change of variables formula to show that X has probability density function

$$h(x) = \frac{1}{\pi \sqrt{1 - x^2}}, \quad -1 < x < 1$$

Note that D and X are not uniformly distributed.

18. Show how to simulate D, A, X, and Y using a random number.

The Model with Uniform Endpoint

Suppose that X is uniformly distributed on the interval $(-1, 1)$.

19. Show that the solution of Bertrand's problem is

$$\mathbb{P}\left(-1 < X < -\frac{1}{2}\right) = \frac{1}{4}$$

20. In Bertrand's experiment, select the uniform endpoint model. Run the experiment 1000 times, updating every 10 runs. Note the apparent convergence of the relative frequency function of the chord event to the true probability.

21. Use the change of variables formula to show that the distance D has density function

$$f(d) = 2\,d, \quad 0 < d < 1$$

22. Use the change of variables formula to show that A has probability density function

$$g(a) = 2 \sin(a) \cos(a), \quad 0 < a < \frac{\pi}{2}$$

Note that D and A are not uniformly distributed; in fact, D has a beta distribution with left parameter 2 and right parameter 1.

Physical Experiments

23. Suppose that a random chord is generated by tossing a coin of radius 1 on a table ruled with parallel lines that are distance 2 apart. Which of the models (if any) would apply to this physical experiment?

24. Suppose that a needle is attached to the edge of disk of radius 1. A random chord is generated by spinning the needle. Which of the models (if any) would apply to this physical experiment?

25. Suppose that a thin trough is constructed on the edge of a disk of radius 1. Rolling a ball in the trough generates a random point on the circle, so a random chord is generated by rolling the ball twice. Which of the models (if any) would apply to this physical experiment?

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