Outside Looking In!
Characterizing an Indecomposable Plane Continuum From Its Complement

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Abstract

It is a long-standing open question whether the Julia set of some rational function is an indecomposable continuum. By definition, a continuum is *indecomposable* iff it is not the union of two of its proper subcontinua. There are several ways of recognizing intrinsically that a continuum X is indecomposable. For instance, X is indecomposable if and only if every proper subcontinuum of X is nowhere dense in X. We provide a condition for testing whether the Julia set of a rational function is an indecomposable continuum using data from its complement. In the complex dynamics context, that means we are investigating the (possibly topologically and dynamically complicated) Julia set from the point of view of the (always topologically and dynamically simple) Fatou set.

The first partial recognition theorem for indecomposable continua from the complement is that of Kuratowski: *If a plane continuum X is the common boundary of three of its complementary domains, then X is either indecomposable or the union of two proper indecomposable subcontinua.* The following classical theorem of Rutt seems quite different, except for its conclusion: *If a nondegenerate plane continuum X is the boundary of a complementary domain U, and if there is a prime end of U whose impression is $\partial U = X$, then X is either indecomposable or the union of two proper indecomposable subcontinua.* The speaker, with various authors (James T. Rogers, Jr., Douglas K. Childers, H. Murat Tuncali, E. D. Tymchatyn), had previously investigated the recognition of indecomposable continua from their complement in the case that $\partial U = X$ for some complementary domain U of X. The tool used was prime end theory. This was applicable to a class of rational functions including all polynomials (of degree ≥ 2). Dynamics, in the case X was a Julia set, was used to rule out "or the union of two proper indecomposable subcontinua" in the above theorems.

A theorem of Burgess covers the case where X is not the boundary of one of its complementary domains.

Burgess's Theorem *If the plane continuum X is the limit of a sequence of distinct complementary domains of X, then either X is indecomposable, or there is only one pair of indecomposable continua whose union is X.*

But it has the same "undesirable" disjunctive conclusion. Moreover, it is an open question whether or not dynamics can rule that out in the absence of the hypothesis that $\partial U = X$ for some complementary domain U.

To state our theorem we need to define some terms. A *generalized crosscut* of a complementary domain U is an open arc $A \subset U$ such that $\overline{A} \setminus A \subset \partial U$. Let U be a plane domain and A a generalized crosscut of U. We call each of the two components of $U \setminus A$ a *crosscut neighborhood.* If V is a crosscut neighborhood determined by generalized crosscut A, we call the continuum $S = \partial V \cap \partial U$ a *shadow* of A. A sequence $(U_n)_{n=1}^{\infty}$ of (not necessarily distinct) complementary domains of a continuum X satisfies the *double-pass condition* iff, for any sequence of generalized crosscuts A_n of U_n, there is a sequence of shadows $(S_n)_{n=1}^{\infty}$ of $(A_n)_{n=1}^{\infty}$ such that $\lim_{n\to\infty} S_n = X$. We prove the following

Characterization Theorem: *A plane continuum X is indecomposable iff X has a sequence $(U_n)_{n=1}^{\infty}$ of complementary domains satisfying the double-pass condition.*

Our results include previous results as special cases. Recently, Clinton Curry has extended Burgess's Theorem and the Characterization Theorem (with an appropriately modified definition of generalized *surface* crosscut) to continua in compact surfaces.

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